The Kadison-Singer Problem in Mathematics and Engineering
Lecture 2: The Paving Conjecture, the $R_\epsilon$-Conjecture, the Bourgain-Tzafriri Conjecture

Master Course on the Kadison-Singer Problem
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In 1979,
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Joel Anderson brought it all back to life.
For $T : \ell^r_2 \to \ell^r_2$, $A \subseteq \{1, 2, \ldots, r\}$

we let $Q_A$ denote the orthogonal projection onto $(e_i)_{i \in A}$. So $Q_A T Q_A$ is the $A \times A$ submatrix of $T$. After a permutation of $\{1, 2, \ldots, r\}$

$$
\begin{bmatrix}
[Q_A T Q_A] & \ldots & .
\vdots & \ddots & \\
\vdots & \ddots & \ddots
\end{bmatrix}
$$
Anderson’s Paving Conjecture

For every $\epsilon > 0$ there exists an $r \in \mathbb{N}$ so that
Paving Conjecture

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For every $\epsilon > 0$ there exists an $r \in \mathbb{N}$ so that

for all $n$ and all $T : \ell^n_2 \to \ell^n_2$ whose matrix has zero diagonal

Important: $r$ depends only on $\epsilon$ and not on $n$ or $T$. 

(Pete Casazza)
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there exists a partition $(A_j)_{j=1}^r$ (called a paving) of $\{1, 2, \ldots, n\}$ so that

\[ \| Q A_j^T Q A_j \| \leq \epsilon \| T \|, \]

for all $j = 1, 2, \ldots, r$. \[\text{Important: } r \text{ depends only on } \epsilon \text{ and not on } n \text{ or } T.\]
## Anderson’s Paving Conjecture

For every $\epsilon > 0$ there exists an $r \in \mathbb{N}$ so that for all $n$ and all $T : \ell_2^n \to \ell_2^n$ whose matrix has zero diagonal there exists a partition $(A_j)_{j=1}^r$ (called a paving) of $\{1, 2, \ldots, n\}$ so that

$$\|Q_{A_j} T Q_{A_j}\| \leq \epsilon \|T\|,$$

for all $j = 1, 2, \ldots, r$. 

$Q_{A_j}$ the orthogonal projection onto span $(e_i)_{i \in A_j}$.
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Important: \( r \) depends only on \( \epsilon \) and not on \( n \) or \( T \).
Pictorially

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\[ r = f(\| T \|, \epsilon). \]
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Then note that there is some $1 \leq j \leq r$ so that for infinitely many $n$, $1 \in A^n_j$. 

CONTINUE!
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Of these infinitely many $n$, there is a $k$ and infinitely many $n$ so that $2 \in A_k^n$. 

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CONTINUE!
Infinite Paving

Infinite Paving Conjecture

Given $\epsilon > 0$ and a bounded operator $T : \ell_2 \to \ell_2$ whose matrix has zero diagonal, there is an $r \in \mathbb{N}$ and a partition $(A_j)_{j=1}^r$ of $\mathbb{N}$ and projections $Q_{A_j}$ so that

$$\|Q_{A_j} T Q_{A_j}\| \leq \epsilon.$$
The Case of Non-Zero Diagonals

Definition

If a matrix $T$ has non-zero diagonal, paving $T$ means to pave it down to the diagonal.

\[ \|QA_jTQ A_j\| \leq (1 + \epsilon) \sup_{i \in I} |T_{ii}|. \]
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$$\| Q_A T Q_A \| \leq (1 + \epsilon) \sup_{i \in I} |T_{ii}|.$$
Paving Operators

To prove the Paving Conjecture it suffices to prove it for any of the following classes of operators:

1. Operators whose matrices have positive coefficients (Halpern, Kaftal, Weiss).

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6. Orthogonal Projections
7. Orthogonal Projections with small diagonal paved to $1 - \epsilon$ (Weaver)
8. Orthogonal Projections on $\ell^2_n$ with constant diagonal $\frac{1}{2}$ (C/Edidin/Kalra/Paulsen)
9. Gram Matrices
10. Lower Triangular matrices (Paulsen/Ragupathi)
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Laurent Operators

If $\phi \in L^\infty[0, 1]$, let

$$T_\phi f = \phi \cdot f \quad \text{for all } f \in L^2[0, 1].$$
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We will look at this in detail later.
Riesable verses Pavable

**Definition**

For \( r \in \mathbb{N} \) and \( 0 < \delta \), an operator \( T \) on \( \mathbb{H}_n \) with \( \| T \| = 1 \) is \((\delta, r)\)-Pavable if

\[
\| Q_{A_j} T Q_{A_j} \| \leq \delta \sum_{i \in A_j} |a_i|^2.
\]

**Definition**

Let \( P \) be a projection on \( \mathbb{H}_n \) with orthonormal basis \( (e_i)_{i=1}^n \).

We say that \((P e_i)\) is \((\delta, r)\)-Riesable if

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\[
\| \sum_{i \in A_j} a_i P e_i \|^2 \geq \delta \sum_{i \in A_j} |a_i|^2.
\]
Let \((e_i)_{i=1}^n\) be an orthonormal basis for \(\mathbb{H}_n\).

[Observation]
Relationship

[Observation]

Let \((e_i)_{i=1}^n\) be an orthonormal basis for \(H_n\). Let \(P\) be an orthogonal projection on \(H_n\), let \(J \subset \{1, 2, \ldots, n\}\) and let

\[
\phi = \sum_{i \in J} a_i e_i
\]

with \(\|\phi\| = 1\).

Then

\[
\|\sum_{i \in J} a_i Pe_i\|_2 \geq \delta > 0 \iff \|(I - P)\phi\|_2 \leq 1 - \delta.
\]
Let \((e_i)_{i=1}^n\) be an orthonormal basis for \(\mathbb{H}_n\). Let \(P\) be an orthogonal projection on \(\mathbb{H}_n\), let \(J \subset \{1, 2, \ldots, n\}\) and let

\[\phi = \sum_{i \in J} a_i e_i, \quad \text{with} \quad \|\phi\| = 1.\]
Let \((e_i)_{i=1}^n\) be an orthonormal basis for \(\mathbb{H}_n\). Let \(P\) be an orthogonal projection on \(\mathbb{H}_n\), let \(J \subset \{1, 2, \ldots, n\}\) and let

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Then\[
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**Theorem**

Let \( (e_i)_{i=1}^n \) be an orthonormal basis for \( \mathbb{H}_n \). Then
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**Theorem**

Let \((e_i)_{i=1}^n\) be an orthonormal basis for \(\mathbb{H}_n\). Then \((Pe_i)_{i=1}^n\) is \((\delta, r)\)-Riesable.
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**Theorem**

Let \((e_i)_{i=1}^n\) be an orthonormal basis for \(\mathbb{H}_n\). Then \((Pe_i)_{i=1}^n\) is \((\delta, r)\)-Riesable

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Gary Weiss reviews the work of Casazza in 2007:
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“Casazza has opened the coffin”
KS in Hilbert Space Theory

Definition

\{ \phi_i \}_{i \in I} \text{ is a Riesz Basic Sequence in } H \text{ if there exist Riesz basis bounds } A, B > 0 \text{ so that for all scalars } (a_i)_{i \in I}

\[ A \sum_{i \in I} |a_i|^2 \leq \left\| \sum_{i \in I} a_i \phi_i \right\|^2 \leq B \sum_{i \in I} |a_i|^2 \]

If \[ a = 1 - \epsilon \], \[ B = 1 + \epsilon \]

This is an \(\epsilon\)-Riesz Basic Sequence

Remark: \((\phi_i)_{i=1}^\infty \) is a Riesz basic sequence if and only if the operator \(T: \ell_2 \to \ell_2 \) given by \(T e_i = \phi_i\) is an invertible operator (on its range) where \((e_i)\) is the unit vector basis of \(\ell_2\).

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Remark:

\((\phi_i)_{i=1}^{\infty}\) is a Riesz basic sequence if and only if the operator \( T : \ell_2 \to \ell_2 \) given by \( Te_i = \phi_i \) is an invertible operator (on its range) where \((e_i)\) is the unit vector basis of \( \ell_2 \).
\( R_\varepsilon \)-Conjecture

For every \( \varepsilon > 0 \), every unit norm Riesz basic sequence is a finite union of \( \varepsilon \)-Riesz Basic Sequences.
Finite-Dimensional $R_{\epsilon}$-Conjecture

For every $\epsilon > 0$ and every invertible $T \in B(\ell^n_2)$ with $\|Te_i\| = 1$ for $i = 1, 2, \ldots, n$, we have

$$(1 - \epsilon) \sum_{i \in A_j} |a_i|^2 \leq \|\sum_{i \in A_j} a_i Te_i\|_2 \leq (1 + \epsilon) \sum_{i \in A_j} |a_i|^2.$$
Finite-Dimensional $R_\varepsilon$-Conjecture

For every $\varepsilon > 0$ and every invertible $T \in B(\ell_2^n)$ with $\|Te_i\| = 1$ for $i = 1, 2, \ldots, n$,

there is an $r = r(\varepsilon, \|T\|, \|T^{-1}\|) \in \mathbb{N}$ and a partition $(A_j)_{j=1}^r$ of $\{1, 2, \ldots, n\}$ so that
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For every $\epsilon > 0$ and every invertible $T \in B(\ell^n_2)$ with $\|Te_i\| = 1$ for $i = 1, 2, \ldots, n$, there is an $r = r(\epsilon, \|T\|, \|T^{-1}\|) \in \mathbb{N}$ and a partition $(A_j)_{j=1}^r$ of $\{1, 2, \ldots, n\}$ so that for all $j = 1, 2, \ldots, r$ and all scalars $(a_i)_{i \in A_j}$ we have
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$$(1 - \epsilon) \sum_{i \in A_j} |a_i|^2 \leq \left\| \sum_{i \in A_j} a_i Te_i \right\|^2 \leq (1 + \epsilon) \sum_{i \in A_j} |a_i|^2.$$
Isomorphisms

**Note:** This form of KS which is not independent of switching to an equivalent norm on KS.
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**Example:**
Define for

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    f = \sum_{i=1}^{\infty} a_i e_i,
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**Example:**
Define for 

\[ f = \sum_{i=1}^{\infty} a_i e_i, \]

\[ |||f||| = \max \left\{ \|f\|_2 + \sup_{1 \leq i} |a_i| \right\}. \]
Theorem

The Paving Conjecture implies the $R_\epsilon$-Conjecture.
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Proof: Given $\epsilon > 0$ and a unit norm Riesz basic sequence $(Te_i)_{i=1}^\infty$ with $\|Te_i\| = 1$, let $S = T^* T$. 
Theorem

The Paving Conjecture implies the $R_\varepsilon$-Conjecture.

Proof: Given $\varepsilon > 0$ and a unit norm Riesz basic sequence $(T e_i)_{i=1}^\infty$ with $\|T e_i\| = 1$, let $S = T^* T$. Note that the diagonal of $S$ is all ones.
Paving and the $R_\epsilon$-Conjecture

**Theorem**

*The Paving Conjecture implies the $R_\epsilon$-Conjecture.*

**Proof:** Given $\epsilon > 0$ and a unit norm Riesz basic sequence $(Te_i)_{i=1}^\infty$ with $\|Te_i\| = 1$, let $S = T^* T$. Note that the diagonal of $S$ is all ones.

By the Paving Conjecture (infinite form) there is an $r \in \mathbb{N}$ and a partition $(A_j)_{j=1}^r$ of $\mathbb{N}$ so that

$$\|Q_{A_j}(I - S)Q_{A_j}\| \leq \delta\|I - S\|,$$

where $\delta = \epsilon/(\|S\| + 1)$. 
Proof Continued

If $\phi = \sum_{i=1}^{\infty} a_i T e_i$, 

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Proof Continued

If \( \phi = \sum_{i=1}^{\infty} a_i T e_i, \)

\[
\left\| \sum_{i \in A_j} a_i T e_i \right\|^2 = \| T Q_{A_j} \|^2
\]
Proof Continued

If \( \phi = \sum_{i=1}^{\infty} a_i Te_i \),

\[
\left\| \sum_{i \in A_j} a_i T e_i \right\|^2 = \| TQ_{A_j} \|^2
\]

= \langle TQ_{A_j} \phi, TQ_{A_j} \phi \rangle

\geq \| Q_{A_j} \phi \|^2 - \delta \| I - S \| \| Q_{A_j} \phi \|^2

\geq (1 - \epsilon) \| Q_{A_j} \phi \|^2

= (1 - \epsilon) \sum_{i \in A_j} |a_i|^2.
If $\phi = \sum_{i=1}^{\infty} a_i T e_i$,

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$$= \langle T Q_{A_j} \phi, T Q_{A_j} \phi \rangle$$

$$= \langle T^* T Q_{A_j} \phi, Q_{A_j} \phi \rangle$$

$$\geq \| Q_{A_j} \phi \|^2 - \delta \| I - S \| \| Q_{A_j} \phi \|^2$$

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$$\geq \| Q A_j \phi \|^2 - \delta \| I - S \| \| Q A_j \phi \|^2$$
Proof Continued

If \( \phi = \sum_{i=1}^{\infty} a_i T e_i \),

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\]

\[
= \langle Q A_j \phi, Q A_j \phi \rangle - \langle Q A_j (I - S) \phi, Q A_j \phi \rangle
\]

\[
\geq \| Q A_j \phi \|^2 - \delta \| I - S \| \| Q A_j \phi \|^2
\]

\[
\geq (1 - \epsilon) \| Q A_j \phi \|^2 = (1 - \epsilon) \sum_{i \in A_j} |a_i|^2.
\]
Restricted Invertibility Theorem

Theorem (Bourgain-Tzafriri Restricted Invertibility Theorem - Spielman and Srivastave form)

For any $0 < \epsilon < 1$ and any natural number $n$, given an orthonormal basis $\{e_i\}_{i=1}^n$ for $\mathbb{H}_n$
Restricted Invertibility Theorem

Theorem (Bourgain-Tzafriri Restricted Invertibility Theorem - Spielman and Srivastave form)

For any $0 < \epsilon < 1$ and any natural number $n$, given an orthonormal basis \( \{e_i\}_{i=1}^n \) for \( \mathbb{H}_n \)
and a bounded linear operator \( L : \mathbb{H}_n \rightarrow \mathbb{H}_n \) with \( \|Le_i\| = 1 \), for all \( i = 1, 2, \ldots, n \),
For any $0 < \epsilon < 1$ and any natural number $n$, given an orthonormal basis \( \{e_i\}_{i=1}^{n} \) for \( \mathbb{H}_n \)

and a bounded linear operator \( L : \mathbb{H}_n \rightarrow \mathbb{H}_n \) with \( \|L e_i\| = 1 \), for all \( i = 1, 2, \ldots, n \),

there is a subset \( I \subset \{ 1, 2, \ldots, n \} \) with

\[
|I| \geq \epsilon^2 \frac{n}{\|L\|^2},
\]
Restricted Invertibility Theorem

Theorem (Bourgain-Tzafriri Restricted Invertibility Theorem - Spielman and Srivastave form)

For any $0 < \epsilon < 1$ and any natural number $n$, given an orthonormal basis $\{e_i\}_{i=1}^n$ for $\mathbb{H}_n$

and a bounded linear operator $L : \mathbb{H}_n \to \mathbb{H}_n$ with $\|Le_i\| = 1$, for all $i = 1, 2, \ldots, n$,

there is a subset $I \subset \{1, 2, \ldots, n\}$ with

$$|I| \geq \epsilon^2 \frac{n}{\|L\|^2},$$

so that for all scalars $\{a_i\}_{i \in I}$ we have

$$(1 - \epsilon)^2 \sum_{i \in I} |a_i|^2 \leq \left\| \sum_{i \in I} a_i Le_i \right\|^2.$$
The Size of Our Subset: $\frac{n}{\|L\|^2}$

Suppose $Le_i = e_1$ for all $i = 1, 2, \ldots, n$. 
The Size of Our Subset: \( \frac{n}{\|L\|^2} \)

Suppose \( Le_i = e_1 \) for all \( i = 1, 2, \ldots, n \).

Then \( \|L\|^2 = n \) and we can only pick one linearly independent vector \( Le_1 \).
The Size of Our Subset: $\frac{n}{\|L\|^2}$

Suppose $Le_i = e_1$ for all $i = 1, 2, \ldots, n$.

Then $\|L\|^2 = n$ and we can only pick one linearly independent vector $Le_1$.

Suppose $Le_{2i} = Le_{2i+1} = e_i$. 
The Size of Our Subset: $\frac{n}{\|L\|^2}$

Suppose $Le_i = e_1$ for all $i = 1, 2, \ldots, n$.

Then $\|L\|^2 = n$ and we can only pick one linearly independent vector $Le_1$.

Suppose $Le_{2i} = Le_{2i+1} = e_i$.

Then $\|L\|^2 = 2$ and we can only pick

$$\frac{n}{\|L\|^2} = \frac{n}{2}$$

vectors.
There exists a universal constant $A > 0$ so that

$$\|Te_i\| = 1 \text{ and } \|T\| \leq B,$$

there exists a partition $(A_j)_{j=1}^r$ of $\{1, 2, \ldots, n\}$ so that for all $j$ and all scalars $(a_i)_{i \in A_j}$

$$\|\sum_{i \in A_j} a_i Te_i\|_2 \geq A \sum_{i \in A_j} |a_i|^2.$$
**KS in Banach Space Theory**

*(strong) Bourgain-Tzafriri Conjecture*

There exists a universal constant $A > 0$ so that for every $0 < B$ there is a natural number $r = r(B)$.
KS in Banach Space Theory

(Strong) Bourgain-Tzafriri Conjecture

There exists a universal constant $A > 0$ so that

for every $0 < B$ there is a natural number $r = r(B)$

so that for every natural number \( n \) and every operator $T : \ell^n_2 \to \ell^n_2$ with $\|Te_i\| = 1$ and $\|T\| \leq B$, \( $\|\sum_{i} a_i Te_i\|_2 \geq A \sum_{i} |a_i|^2$ \)
**KS in Banach Space Theory**

*(strong) Bourgain-Tzafriri Conjecture*

There exists a universal constant $A > 0$ so that

for every $0 < B$ there is a natural number $r = r(B)$

so that for every natural number "n" and every operator $T : \ell^n \to \ell^n$ with $\|Te_i\| = 1$ and $\|T\| \leq B$,

there exists a partition $(A_j)_{j=1}^r$ of $\{1, 2, \ldots, n\}$ so that for all $j$ and all scalars $(a_i)_{i \in A_j}$

$$\left\| \sum_{i \in A_j} a_i Te_i \right\|^2 \geq A \sum_{i \in A_j} |a_i|^2$$
(strong) Bourgain-Tzafriri Conjecture

There exists a universal constant $A > 0$ so that for every $0 < B$ there is a natural number $r = r(B)$ so that for every natural number "n" and every operator $T : \ell^n_2 \to \ell^n_2$ with $\|Te_i\| = 1$ and $\|T\| \leq B$,

there exists a partition $(A_j)_{j=1}^r$ of $\{1, 2, \ldots, n\}$ so that for all $j$ and all scalars $(a_i)_{i \in A_j}$

$$\left\| \sum_{i \in A_j} a_i Te_i \right\|^2 \geq A \sum_{i \in A_j} |a_i|^2$$

(weak) Bourgain-Tzafriri Conjecture

$A = f(B)$
The $R_\epsilon$-Conjecture implies the Bourgain-Tzafriri Conjecture.

**Proof:** If $\|T e_i\| = 1$ for all $i = 1, 2, \ldots$, in $\ell_2 \oplus \ell_2$ let 
$$\phi_i = (\sqrt{1 - \epsilon^2} T e_i, \epsilon e_i).$$
The $R_\epsilon$-Conjecture implies the Bourgain-Tzafriri Conjecture.

**Proof:** If $\|T e_i\| = 1$ for all $i = 1, 2, \ldots$, in $\ell_2 \oplus \ell_2$ let 
\[ \phi_i = (\sqrt{1 - \epsilon^2} T e_i, \epsilon e_i). \]
Then $\|\phi_i\| = 1$ and $(\phi_i)$ is a Riesz basic sequence.
The $R_\epsilon$-Conjecture implies the Bourgain-Tzafriri Conjecture.

**Proof:** If $\|Te_i\| = 1$ for all $i = 1, 2, \ldots$, in $\ell_2 \oplus \ell_2$ let 
$\phi_i = (\sqrt{1 - \epsilon^2} Te_i, \epsilon e_i)$.

Then $\|\phi_i\| = 1$ and $(\phi_i)$ is a Riesz basic sequence.

So we can partition $\mathbb{N}$ into $(A_j)_{j=1}^r$ so that for all $j = 1, 2, \ldots, r$ and all $(a_i)_{i \in A_j} \in \ell_2$ we have
Proof Continued

\[(1 - \epsilon^2) \sum_{i \in A_j} |a_i|^2 \leq \| \sum_{i \in A_j} a_i \phi_i \|^2 \]
Proof Continued

\[
(1 - \epsilon^2) \sum_{i \in A_j} |a_i|^2 \leq \| \sum_{i \in A_j} a_i \phi_i \|^2 \\
= (1 - \epsilon^2) \| \sum_{i \in A_j} a_i T e_i \|^2 + \epsilon^2 \sum_{i \in A_j} |a_i|^2
\]
Proof Continued

\[(1 - \epsilon^2) \sum_{i \in A_j} |a_i|^2 \leq \| \sum_{i \in A_j} a_i \phi_i \|^2 \]

\[= (1 - \epsilon^2) \| \sum_{i \in A_j} a_i T e_i \|^2 + \epsilon^2 \sum_{i \in A_j} |a_i|^2 \]

Hence,

\[\frac{1 - 2\epsilon^2}{1 - \epsilon^2} \sum_{i \in A_j} |a_i|^2 \leq \| \sum_{i \in A_j} a_i T e_i \|^2.\]
Our Tour of the Kadison-Singer Problem

Marcus/Spielman/Srivastava ⇒ Casazza/Tremain Conjecture and Weaver Conjecture $KS_r$
⇒ Weaver Conjecture
⇒ Paving Conjecture
⇒ $R_\epsilon$-Conjecture
⇒ Bourgain-Tzafriri Conjecture
⇒ Feichtinger Conjecture
⇒ Sundberg Problem

Finally:

Bourgain-Tzafriri Conjecture ⇒ Weaver Conjecture $KS_r$
⇒ Paving Conjecture
⇔ The Kadison-Singer Problem