The Kadison-Singer Problem in Mathematics and Engineering
Lecture 2: The Paving Conjecture, the R_ϵ-Conjecture, the Bourgain-Tzafriri Conjecture

Master Course on the Kadison-Singer Problem
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Joel Anderson brought it all back to life.

KS in Operator Theory

Notation

$$For \quad T : \ell_2^r \to \ell_2^r \quad A \subseteq \{1, 2, \ldots, r\}$$

we let Q_A denote the orthogonal projection onto $(e_i)_{i \in A}$. So $Q_A T Q_A$ is the $A \times A$ submatrix of T. After a permutation of $\{1, 2, \ldots, r\}$

$$\begin{array}{cc} & A \\ A & \begin{bmatrix} [Q_A T Q_A] \ldots & \ldots \\ \vdots \qquad \ldots & \ldots \\ \vdots \qquad \vdots & \ldots \\ \vdots \qquad \vdots & \vdots \\ & \end{bmatrix} \end{array}$$

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$$\|Q_{A_j} T Q_{A_j}\| \le \epsilon \|T\|, \quad \text{for all } j = 1, 2, \ldots, r.$$

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Important: r depends only on ϵ and not on n or T.

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CONTINUE!

Infinite Paving

Infinite Paving Conjecture

Given $\epsilon > 0$ and a bounded operator $T : \ell_2 \to \ell_2$ whose matrix has zero diagonal, there is an $r \in \mathbb{N}$ and a partition $(A_j)_{j=1}^r$ of $\mathbb{N}$ and projections Q_{A_j} so that

$$\|Q_{A_j} T Q_{A_j}\| \leq \epsilon.$$

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If a matrix T has non-zero diagonal, paving T means to pave it down to the diagonal.

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$$\|Q_{A_j}TQ_{A_j}\| \leq (1+\epsilon)\sup_{i\in I}|T_{ii}|.$$

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To prove the Paving Conjecture it suffices to prove it for any of the following classes of operators:

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10. Lower Triangular matrices (Paulsen/Ragupathi)

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We will look at this in detail later.

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$$\|\sum_{i \in A_j} a_i P e_i\|^2 \geq \delta \sum_{i \in A_j} |a_i|^2.$$

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$$\| \sum_{i \in J} a_i P e_i \|^2 \geq \delta > 0 \Leftrightarrow \|(I - P)\phi\|^2 \leq 1 - \delta.$$

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Gary Weiss reviews the work of Casazza in 2007:

> "Casazza has opened the coffin"

KS in Hilbert Space Theory

Definition

$\{\phi_i\}_{i \in I}$ is a Riesz Basic Sequence in H if there exist Riesz basis bounds $A, B > 0$ so that for all scalars $(a_i)_{i \in I}$

$$A \sum_{i \in I} |a_i|^2 \leq \left\| \sum_{i \in I} a_i \phi_i \right\|^2 \leq B \sum_{i \in I} |a_i|^2$$

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Remark:

$(\phi_i)_{i=1}^{\infty}$ is a Riesz basic sequence if and only if the operator $T : \ell_2 \to \ell_2$ given by $Te_i = \phi_i$ is an invertible operator (on its range) where (e_i) is the unit vector basis of ℓ_2.

C/Vershynin Conjecture

R_ϵ-Conjecture

For every $\epsilon > 0$, every unit norm Riesz basic sequence is a finite union of ϵ-Riesz Basic Sequences.

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$$(1-\epsilon)\sum_{i \in A_j} |a_i|^2 \le \|\sum_{i \in A_j} a_i Te_i\|^2 \le (1+\epsilon)\sum_{i \in A_j} |a_i|^2.$$

Isomorphisms

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$$|||f||| = \max \left\{ \|f\|_2 + \sup_{1 \le i} |a_i| \right\}.$$

Paving and the R_ϵ-Conjecture

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The Paving Conjecture implies the R_ϵ-Conjecture.

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Proof: Given $\epsilon > 0$ and a unit norm Riesz basic sequence $(Te_i)_{i=1}^{\infty}$ with $\|Te_i\| = 1$, let $S = T^*T$.

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Proof: Given $\epsilon > 0$ and a unit norm Riesz basic sequence $(Te_i)_{i=1}^{\infty}$ with $\|Te_i\| = 1$, let $S = T^*T$. Note that the diagonal of S is all ones.

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Proof: Given $\epsilon > 0$ and a unit norm Riesz basic sequence $(Te_i)_{i=1}^{\infty}$ with $\|Te_i\| = 1$, let $S = T^*T$. Note that the diagonal of S is all ones.

By the Paving Conjecture (infinite form) there is an $r \in \mathbb{N}$ and a partition $(A_j)_{j=1}^{r}$ of $\mathbb{N}$ so that

$$\|Q_{A_j}(I - S)Q_{A_j}\| \leq \delta\|I - S\|,$$

where $\delta = \epsilon/(\|S\| + 1)$.

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$$\begin{aligned} \left\| \sum_{i \in A_j} a_i Te_i \right\|^2 &= \| TQ_{A_j} \|^2 \\ &= \langle TQ_{A_j}\phi, TQ_{A_j}\phi \rangle \end{aligned}$$

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&= \langle Q_{A_j} \phi, Q_{A_j} \phi \rangle - \langle Q_{A_j} (I - S) \phi, Q_{A_j} \phi \rangle
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&= \langle Q_{A_j}\phi, Q_{A_j}\phi \rangle - \langle Q_{A_j}(I - S)\phi, Q_{A_j}\phi \rangle \\
&\geq \|Q_{A_j}\phi\|^2 - \delta \|I - S\| \|Q_{A_j}\phi\|^2
\end{aligned}$$

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&\geq \|Q_{A_j}\phi\|^2 - \delta \|I - S\| \|Q_{A_j}\phi\|^2 \\
&\geq (1 - \epsilon)\|Q_{A_j}\phi\|^2 = (1 - \epsilon) \sum_{i \in A_j} |a_i|^2.
\end{aligned}
$$

Restricted Invertibility Theorem

Theorem (Bourgain-Tzafriri Restricted Invertibility Theorem - Spielman and Srivastave form)

For any $0 < \epsilon < 1$ *and any natural number n, given an orthonormal basis* $\{e_i\}_{i=1}^n$ *for* $\mathbb{H}_n$

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there is a subset $I \subset \{1, 2, \ldots, n\}$ with

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so that for all scalars $\{a_i\}_{i \in I}$ we have

$$(1-\epsilon)^2 \sum_{i \in I} |a_i|^2 \leq \|\sum_{i \in I} a_i L e_i\|^2.$$

The Size of Our Subset: $\frac{n}{\|L\|^2}$

Suppose $Le_i = e_1$ for all $i = 1, 2, \ldots, n$.

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Then $\|L\|^2 = 2$ and we can only pick

$$\frac{n}{\|L\|^2} = \frac{n}{2} \text{ vectors.}$$

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there exists a partition $(A_j)_{j=1}^r$ of $\{1, 2, \ldots, n\}$ so that for all j and all scalars $(a_i)_{i \in A_j}$

$$\left\| \sum_{i \in A_j} a_i Te_i \right\|^2 \geq A \sum_{i \in A_j} |a_i|^2$$

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(weak) Bourgain-Tzafriri Conjecture

$A = f(B)$

R_ϵ-Conjecture and BT

Theorem

The R_ϵ-Conjecture implies the Bourgain-Tzafriri Conjecture.

Proof: If $\|Te_i\| = 1$ for all $i = 1, 2, \ldots$, in $\ell_2 \oplus \ell_2$ let $\phi_i = (\sqrt{1-\epsilon^2}\, Te_i, \epsilon e_i)$.

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$\phi_i = (\sqrt{1-\epsilon^2}\,Te_i, \epsilon e_i)$.
Then $\|\phi_i\| = 1$ and (ϕ_i) is a Riesz basic sequence.
So we can partition $\mathbb{N}$ into $(A_j)_{j=1}^r$ so that for all $j = 1, 2, \ldots, r$ and all $(a_i)_{i \in A_j}$ we have

Proof Continued

$$(1-\epsilon^2)\sum_{i\in A_j}|a_i|^2 \quad \le \quad \|\sum_{i\in A_j} a_i\phi_i\|^2$$

Proof Continued

$$
\begin{aligned}
(1-\epsilon^2)\sum_{i\in A_j}|a_i|^2 &\le \|\sum_{i\in A_j} a_i\phi_i\|^2 \\
&= (1-\epsilon^2)\|\sum_{i\in A_j} a_i Te_i\|^2 + \epsilon^2 \sum_{i\in A_j}|a_i|^2
\end{aligned}
$$

Proof Continued

$$\begin{aligned}(1-\epsilon^2)\sum_{i\in A_j}|a_i|^2 &\leq \|\sum_{i\in A_j} a_i\phi_i\|^2 \\ &= (1-\epsilon^2)\|\sum_{i\in A_j} a_i Te_i\|^2 + \epsilon^2\sum_{i\in A_j}|a_i|^2\end{aligned}$$

Hence,

$$\frac{1-2\epsilon^2}{1-\epsilon^2}\sum_{i\in A_j}|a_i|^2 \leq \|\sum_{i\in A_j} a_i Te_i\|^2.$$

Our Tour of the Kadison-Singer Problem

Marcus/Spielman/Srivastava	$\Rightarrow$	Casazza/Tremain Conjecture
	and	Weaver Conjecture KS_r
	$\Rightarrow$	Weaver Conjecture
	$\Rightarrow$	Paving Conjecture
	$\Rightarrow$	R_ϵ-Conjecture
	$\Rightarrow$	Bourgain-Tzafriri Conjecture
	$\Rightarrow$	Feichtinger Conjecture
	$\Rightarrow$	Sundberg Problem

Finally:

Bourgain-Tzafriri Conjecture	$\Rightarrow$	Weaver Conjecture KS_r
	$\Rightarrow$	Paving Conjecture
	$\Leftrightarrow$	The Kadison-Singer Problem