The Kadison-Singer Problem in Mathematics and Engineering
Lecture 1: The Kadison-Singer Problem
An Introduction to Frame Theory

Master Course on the Kadison-Singer Problem
University of Copenhagen

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October 14, 2013

Supported By

The Defense Threat Reduction Agency

NSF-DMS

The National Geospatial Intelligence Agency.

The Air Force Office of Scientific Research

Outline of the Areas We Will Touch on This Week

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Our Tour of the Kadison-Singer Problem

Marcus/Spielman/Srivastava $\Rightarrow$ Casazza/Tremain Conjecture

and Weaver Conjecture KS_r

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Dirac's Book

P.A.M. Dirac (1947)
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"Quantum Mechanics"

Dirac wants to find a representation (i.e., an orthonormal basis) for a compatible family of observables (i.e. a commuting family of self-adjoint operators). On pages 74-75 he writes:

Dirac's Book

"To introduce a representation in practice

(i) We look for observables which we would like to have diagonal either because we are interested in their probabilities or for reasons of mathematical simplicity;

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The representation is then completely determined ...by the observables that are diagonal..."

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Pure State

Extreme point in the set of states.

Dirac's Mistake

Dirac's Claim

W_{e_i} has a unique extension to a linear functional on $B(H)$.

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His Mistake

There are many other pure states on D.

The 1959 Kadison-Singer Problem

Kadison-Singer Problem

Does every Pure State on $D =$ family of diagonal operators on H extend uniquely to a pure state on $B(H)$?

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2. Even Industrial Applications of frames often lead to fundamental questions in both pure and applied mathematics

Frame Theory uses a wide variety of tools

- Functional Analysis
- C^*-algebras
- Operator Theory
- Banach Space Theory
- Probability
- Graph Theory
- Combinatorics
- Algebraic Geometry
- Number theory...

Parseval's Identity

Parseval's Identity

If (e_n) is an orthonormal basis for a Hilbert space H, then for all $\phi \in H$:

$$\|\phi\|^2 = \sum_n |\langle \phi, e_n \rangle|^2$$

Moreover:

$$\phi = \sum_n \langle \phi, e_n \rangle e_n$$

Many other sets of vectors satisfy Parseval's identity

If $(e_n)_{n\in I}$, $(\phi_n)_{n\in I}$ are orthonormal bases for H then

$$\left(\frac{1}{\sqrt{2}}e_n\right) \cup \left(\frac{1}{\sqrt{2}}\phi_n\right) \tag{1}$$

satisfies Parseval's identity.

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And:

$$\begin{array}{ccccccc} e_1 & 0 & \frac{1}{\sqrt{2}}e_2 & 0 & \frac{1}{\sqrt{3}}e_3 & 0 & \ldots \\ & 0 & \frac{1}{\sqrt{2}}e_2 & 0 & \frac{1}{\sqrt{3}}e_3 & 0 & \ldots \\ & & & & \frac{1}{\sqrt{3}}e_3 & 0 & \ldots \end{array} \tag{2}$$

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Duffin and Schaeffer 1952

$(\phi_i)_{i \in I}$ is a frame for a (finite or infinite dimensional) Hilbert space $\mathbb{H}$ if there exists $A, B > 0$ so that for every $\phi \in \mathbb{H}$

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$\|\phi_i\| = 1$ Unit Norm Frame.

Frame Bounds and Projections

Remark

If (ϕ_i) is a frame for $\mathbb{H}$ with frame bounds A, B and P is an orthogonal projection on H then$(P\phi_i)$ is a frame for $P\mathbb{H}$ with frame bounds A, B.

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Remark

The Bessel bound of (ϕ_i) is just the norm of the operator

$$Te_i = \phi_i,$$

where $(e_i)_{i \in I}$ is an orthonormal basis for $\mathbb{H}$.

Analysis Operator of the Frame

If $(\phi_i)_{i\in I}$ is a sequence of vectors in a Hilbert space $\mathbb{H}$ we define the analysis operator $T : \mathbb{H} \to \ell_2(I)$ by:

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Parseval Frame if and only if T is an isometry (onto its range).

The Synthesis Operator and the Frame Operator

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The frame operator is $S = T^*T : H \rightarrow H$ where $S\phi = \sum \langle \phi, \phi_i \rangle \phi_i$.

Hence,

$$\langle S\phi, \phi \rangle = \sum |\langle \phi, \phi_i \rangle|^2$$

We Love Parseval Frames

So S is a positive, self-adjoint invertible operator on $\mathbb{H}$ satisfying

$$A \cdot I \leq S \leq B \cdot I$$

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if and only if

$$\|\phi\|^2 = \sum \langle \phi, \phi_i \rangle |^2, \text{ for all } \phi \in \mathbb{H}.$$

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So, for any frame $\{\phi_i\}$, $\{S^{-\frac{1}{2}}\phi_i\}$ is a Parseval frame.

Classifying Parseval Frames

Theorem

(ϕ_i) is a Parseval frame for H if and only if there is a containing Hilbert space $\mathbb{H} \subset \mathbb{K}$ with an orthonormal basis (e_i) and the orthogonal projection P of $\mathbb{K}$ onto $\mathbb{H}$ satisfies:

Classifying Parseval Frames

Theorem

(ϕ_i) *is a Parseval frame for H if and only if there is a containing Hilbert space* $\mathbb{H} \subset \mathbb{K}$ *with an orthonormal basis* (e_i) *and the orthogonal projection* P *of* $\mathbb{K}$ *onto* $\mathbb{H}$ *satisfies:*

$$Pe_i = \phi_i, \text{ for all } i.$$

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$$\langle T\phi_m, Pe_i\rangle = \langle PT\phi_m, e_i\rangle$$

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$\Rightarrow$ Given a Parseval frame (ϕ_i) with analysis operator T (which is an isometry) we have

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Since $(T\phi_m)$ spans $P\mathbb{H}$, it follows that

$$Pe_i = T\phi_i \text{ for all } i.$$

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Corollary

(ϕ_i) is a Parseval frame if and only if there is a Parseval frame (ψ_i) so that $(\phi_i \oplus \psi_i)$ is an orthonormal sequence.

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4. An ability to capture important signal characteristics.