Characterization of locally reflexive algebras
by an “inner” version of exactness

Eberhard Kirchberg

HU Berlin

Copenhagen, Nov 2011
Some Notation

Let $A$ and $B$ C*-algebras.

$\| \cdot \|_\infty = \| \cdot \|_{\min}$ means the spatial tensor product norm on the algebraic tensor product $A^{**} \circ B^{**}$,

We denote $A^{**} \otimes B^{**}$ the the C*-algebra completion of $A^{**} \circ B^{**}$ with $\| \cdot \|$.

Let $X \subset \mathcal{L}(H_1)$ and $Y \subset \mathcal{L}(H_2)$ closed subspaces (operator spaces). $X \otimes Y$ is the operator space that is the closure of vector space tensor product $X \circ Y$ in $\mathcal{L}(H_1 \otimes H_2)$. 
Locally reflexive C*-algebras

Let $X \subset \mathcal{L}(H)$ a unital linear subspace of finite dimension, and $V : X \to A^{**}$ a unital completely contractive map.

In general, it is not possible to find a family $\{V_\gamma\}$ of completely contractive maps $V_\gamma : X \to A$, such that $V$ is the point-$\sigma(A^{**}, A^*)$ limit of the $V_\gamma$.

**Definition (1)**

A is **locally reflexive** (in a matricial sense) if, for every subspaces $X \subset A^{**}$ and $F \subset A^*$ of finite dimension and for every $\varepsilon > 0$ there exists a completely contractive linear map $T : X \to A$ with

$$|x(f) - f(T(x))| \leq \varepsilon \|x\| \cdot \|f\|. \quad \forall x \in X, f \in F.$$  

$\mathcal{L}(\ell_2)$ and $C^*(SL(\mathbb{Z}))$ are not locally reflexive.
The C-norm $\| \cdot \|_C$ on $A^{**} \odot B^{**}$

The algebraic tensor product $A^{**} \odot B^{**}$ is a $^*$-subalgebra of $(A \otimes B)^{**}$ in a natural way. The induced $C^*$-norms on $A^{**} \odot B^{**}$, $A \odot B^{**}$ or $A^{**} \otimes B$ will be denoted by $\| \cdot \|_C$.

**Lemma (2)**

For $a_1, \ldots, a_n \in A^{**}$, $b_1, \ldots, b_n \in B^{**}$, and $w = \sum_k a_k \otimes b_k$ holds

$$\| w^* w \|_C = (\| w \|_C)^2 = \sup_{\lambda} \sum_{j,k} \lambda((a_j^* a_k) \otimes (b_j^* b_k)) = \sup_{\lambda} \lambda(w^* w)$$

where $\lambda$ runs over all unital positive linear functionals $\lambda$ on $A^{**} \odot B^{**}$ that are partially normal on $A^{**}$ and on $B^{**}$, and are continuous on $A \odot B$ with respect to $\| \cdot \|_{\min}$ on $A \odot B$. 
It means, that \( \lambda(a \otimes b) = \langle d_1(a)d_2(b)x, x \rangle \) where \( d_1 : A^{**} \to \mathcal{L}(H) \) and \( d_2 : B^{**} \to \mathcal{L}(H) \) are commuting non-degenerate (= unital) normal *-representations of the von Neumann algebras \( A^{**} \) and \( B^{**} \), and that the *-representation

\[
\sum_k a_k \otimes b_k \mapsto \sum_k d_1(a_k)d_2(b_k)
\]

is continuous on \( A \odot B \) with respect to the minimal C*-norm on \( A \odot B \), and the vector \( x \in H \) has norm \( \|x\| = 1 \).
Lemma (3) (Reduction to separable case)

Suppose that $X \subset A$, $Y \subset A^*$ and $Z \subset A^{**}$ are (norm-)separable subspaces. Then there exist a separable $C^*$-subalgebra $B$ of $A$ and a normal completely positive map $V : A^{**} \to B^{**} \cong B^{\text{strong}}$

and a projection $P \in V(A^{**})' \cap B^{**}$ such that

1. $X \subset B$ and $V(x) = x$ for all $x \in C^*(X)$,

2. $\rho(V(a)) = \rho(a)$ for all $a \in A^{**}$ and $\rho \in Y$, i.e., $V_*$ fixes $Y$,

3. $a \mapsto V(a)P$ is multiplicative on $C^*(X \cup Z)$, and $P$ is countably decomposable (in $B^{**}$),

4. $Py = y = yP$ for all $y \in Y$. 

Eberhard Kirchberg (HU Berlin)  Characterization of locally reflexive algebras  Copenhagen, Nov 2011  7 / 32
In particular, the separable $C^*$-subalgebra $B \subset A$ and the $C^*$-morphism 
$\phi: C^*(Z \cup X) \to B^{**} \subset A^{**}$ given by $\phi(a) := V(a)P$ satisfy 
$X \subset B$, $\rho(\phi(z)) = \rho(z)$ for $\rho \in Y$, $z \in Z$, and $\phi(x) = x$ for $x \in X$.

It yields (the non-trivial part of the proof of):

**Lemma (4)**

\[
\| \cdot \|_C = \| \cdot \|_{\min \text{ on } A^{**} \odot B}, \quad \text{if and only if,}
\]
\[
\| \cdot \|_C = \| \cdot \|_{\min \text{ on } D^{**} \odot E \text{ for all separable } C^*-\text{subalgebras } D \subset A \text{ and } E \subset B},
\]
\[
\quad \text{if and only if,}
\]
\[
\| \cdot \|_C = \| \cdot \|_{\min \text{ on } (K \otimes A)^{**} \odot B}, \quad \text{if and only if,}
\]
\[
\| \cdot \|_C = \| \cdot \|_{\min \text{ on } (A + \mathbb{C} \cdot 1)^{**} \odot B}.
\]

Notice that $\| \cdot \|_C = \| \cdot \|$ on $A \odot B$. 
Let $L \subset A$ a closed left-ideal and $p_L \in A^{**}$ its open support projection, and define linear $\pi : A \otimes B \to (A/L) \otimes B \subset (A/L) \otimes_{\min} B$ by

$$\pi(a \otimes b) := (a + L) \otimes b.$$ 

**Lemma (5)**

*If $w \in A \otimes B$ then $w((1 - p_L) \otimes 1) \in A^{**} \otimes B$,

$$\text{dist}(w, L \otimes_{\min} B) = \text{dist}(w, L \otimes B) = \|w((1 - p_L) \otimes 1)\|_C$$

and

$$\|\pi(w)\| = \|w((1 - p_L) \otimes 1)\|_{\min}.$$*
Properties (C), (C’) and (C’’)

**Definition (6)**

The algebra $A$ has property

$(C)$ if $\| \cdot \|_C = \| \cdot \|_{min}$ on $A^{**} \odot B^{**}$ for every $C^*$-algebra $B$,

$(C’)$ if $\| \cdot \|_C = \| \cdot \|_{min}$ on $A \odot B^{**}$ for every $C^*$-algebra $B$,

$(C’’)$ if $\| \cdot \|_C = \| \cdot \|_{min}$ on $A^{**} \odot B$ for every $C^*$-algebra $B$.

Effros and Haagerup: All this properties pass to subalgebras, $E \subset A$, and properties $(C)$ and $(C’’)$ pass to quotients $A/J$.

$A$ has property $(C’’)$ if and only if $A$ is locally reflexive.

E.K.(in Crelle J.): $(C’) \Rightarrow$ exactness $\Rightarrow (C)$. In particular, $(C) = (C’)$.
It is known:

- Each exact C*-algebra is locally reflexive.
- A C*-algebra is locally reflexive, if and only if, all its separable C*-subalgebras are locally reflexive.
- Locally reflexive C*-algebras with WEP (of Lance) are nuclear.
- Locally reflexive C*-algebras with a matricial variant of the Grothendieck approximation property are exact.
- Extensions of locally reflexive C*-algebras are locally reflexive, if and only if, the Busby invariant is locally liftable.
- Locally reflexive algebra $A$ is exact, if and only if, $A^{**}$ is a weakly exact $W^*$-algebra.
Open problems concerning local reflexivity:

Let $A, A_1 \subset A_2 \subset \cdots$ locally reflexive (=: l.r.) $C^*$-algebras.

(a) Is $A$ exact? In particular: Let $G$ a Gromov example (=: discrete finitely presented group that is not uniformly embeddable into a Hilbert space with respect to its word length metric). Is $C^*_r(G)$ not l.r.? (It is not exact by a result of Ozawa.)

(b) Is $M_{2\infty} \otimes A$ l.r.? (Equivalent to: Is $B \otimes A$ l.r. if $B$ is exact?)

(c) Are inductive limits of l.r. algebras $A_n$ again l.r.?

(d) Is the crossed product $A \rtimes_\alpha \mathbb{Z}$ of $A$ by $\alpha \in \text{Aut}(A)$ again l.r.?

(f) Are reduced free products of l.r. algebras again l.r.?

(g) Suppose $A$ does not have the WEP. Are there states $\lambda$ on $A$ and $\mu$ on $C[0, 1]$ such that $A * 1 \subset A *_{\rho, \mu} C[0, 1]$ is not relatively weakly injective in the reduced free product $A *_{\rho, \mu} C[0, 1]$?

(h) Is $A$ embeddable into simple l.r. $C^*$-algebra $B$?
Possible positive answers to questions (a – h) have the following implications:

(c) $\Rightarrow$ (b),
(d) $\Rightarrow$ (b, f),
(a) $\Rightarrow$ (b – f, h),
(d) $\Rightarrow$ (f),
(c, f, g) $\Rightarrow$ (a),
(f) $\Rightarrow$ (h).
Definition (7)

We call a $C^*$-algebra $A$ **inner exact for** $B$, if the sequence

$$0 \to L \otimes B \to (\mathcal{K} \otimes A) \otimes B \to ((\mathcal{K} \otimes A)/L) \otimes B \to 0$$

is exact (in the metric sense) for every closed left-ideal $L \subset \mathcal{K} \otimes A$. The algebra $A$ is **inner exact** if $A$ is inner exact for every $C^*$-algebra $B$.

By Lemma 5, $\| \cdot \|_C = \| \cdot \|_{\min}$ on $A^{**} \otimes B$ implies that $A$ is *inner exact* for $B$.

$A$ is inner exact $\iff \forall \ X \subset (\mathcal{K} \otimes A)/L$ with $\text{Dim}(X) < \infty$ and $\varepsilon > 0$,

$\exists \ T : X \to \mathcal{K} \otimes A$ with $\pi_L \circ T = \text{id}_X$ and $\| T \|_{cb} \leq 1 + \varepsilon$. 
Reduction to the separable and unital case

**Lemma (8)**

**TFAE:**

- $A$ is inner exact for $B$.
- Every separable $C^*$-subalgebra of $A$ is inner exact for $B$.
- $K \otimes A$ is inner exact for $B$.
- The unitization $\tilde{A}$ is inner exact for $B$.

Since the same happens for the property that $\| \cdot \|_C = \| \cdot \|_{\min}$ on $A^{**} \otimes B$, it suffices to prove the following Proposition 9 only for separable unital $A$. 
Proposition (9)

\[ \| \cdot \|_C = \| \cdot \|_{\min} \text{ on } A^{**} \otimes B, \text{ if and only if, } A \text{ is inner exact for } B. \]

If we combine this with the work of Effros and Haagerup, we get:

Theorem (10) (characterization of locally reflexive algebras)

\( A \) C*-algebra \( A \) is locally reflexive, if and only if, the sequence of operator spaces

\[
0 \to L \otimes B \to (K \otimes A) \otimes B \to ((K \otimes A)/L) \otimes B \to 0
\]

is exact – in the complete metric sense – for every closed left-ideal \( L \) of \( K \otimes A \) and every C*-algebra \( B \).
It is an open question if every locally reflexive algebra is exact. If this question would have a positive answer, then we could get from Theorem 10, or even better from a proof of the following Conjecture 11, an “algebraic” inner characterization of exactness.

**Conjecture (11)**

If, for every C*-subalgebra $E \subset A$ and every closed ideal $J$ of $E$ the Busby invariant of the extension $0 \to J \to E \to E/J \to 0$ is locally liftable, then $A$ is locally reflexive.
On the Proof of Proposition 9:
Lemmata 4 and 8 show that it suffices to consider the case of separable and unital $A$.

Proof of “$\Rightarrow$” follows from Lemmata 4 and 5.

The proof of “$\Leftarrow$” needs some ideas related to the nc Lusin Theorem and to desired non-commutative versions of the Egorov theorem.

Lemma 2, repeated use of the nc Lusin Theorem (M.Tomita 1959, see book of G.K. Pedersen, Thm. 2.7.3), and $N(x^*x) = N(xx^*)$ for each $C^*$-norm $N$ on $A^{**} \odot B$ together prove the following lemma.

**Lemma (12)**

If $\|v^*(p \otimes 1)v\|_C \leq \|v^*(p \otimes 1)v\|_{\text{min}}$ for all $v \in A \odot B$ and all countably decomposable projections $p \in A^{**}$, then $\| \cdot \|_C = \| \cdot \|_{\text{min}}$ on $A^{**} \odot B$. 

Eberhard Kirchberg (HU Berlin) Characterization of locally reflexive algebras Copenhagen, Nov 2011 18 / 32
Lemmata 12 and Lemma 2 show that $\| \cdot \|_C = \| \cdot \|_{\text{min}}$ on $A^{**} \odot B$, if for each (fixed) $v \in A \odot B$ and each (fixed) positive partially normal state $\lambda$ on $A^{**} \odot B^{**}$, that is continuous on $A \odot B$ with respect to $\| \cdot \|_{\text{min}}$, holds

$$\lambda(\nu^*(p \otimes 1)v) \leq \|\nu^*(p \otimes 1)v\|_{\text{min}}$$

for all countably decomposable projection $p \in A^{**}$. 
Fix $v \in A \odot B$, $\lambda$ as above. Find countably decomposable projection $c$ in the center of $A^{**}$ with $pc = p$, $\lambda(c \otimes 1) = 1$, and $\|ac\| = \|a\|$ for all $a \in A$. Then $A^{**}c$ has a faithful normal unital representation on a separable Hilbert space. Thus, the Up-Down Theorem of G.K. Pedersen applies. Since $A \cong Ac \subset A^{**}c$ is unital, we get that each element of $A^{**}_+$ with norm $\leq 1$ (in particular our projection $p$) is in $((A^1_+)_{\sigma-\text{down}})_{\sigma-\text{up}}$.

We define the set $S = S(A, \lambda, v)$ of $a \in A^{**}$ with $0 \leq a \leq 1$ and the property and $\lambda(v^*(a \otimes 1)v) \leq \|v^*(ac \otimes 1)v\|_{\text{min}}$.

Since $a \in A^{**} \mapsto \|v^*(a \otimes 1)v\|$ is order preserving, and since $a \in A^{**} \mapsto \lambda(v^*(a \otimes 1)v)$ is a normal positive functional, we get that $a \in S$ if $a = \sup_n a_n$ for $a_1 \leq a_2 \leq \cdots \in S$. 
Intermediate result:
If $\lambda(v^*(a \otimes 1)v) \leq \|v^*((ca) \otimes 1)v\|_{\min}$ with $c$ as above selected (depending on $\lambda$ and $v$) for all $a \in (A_+^{-1})_{\sigma-\text{down}}$, each $\lambda$ and $v \in A \odot B$, then $\| \cdot \|_C = \| \cdot \|_{\min}$ on $A^{**} \odot B$.

The proof of Proposition 9 becomes complete, if we can find a closed projection $q \in (K \otimes A)^{**}$ such that $e_{11} \otimes (v^*(a \otimes 1)v) = (e_{11} \otimes v)(q \otimes 1)(e_{11} \otimes v)$. 
Let $A$ a unital C*-algebra and $0 \leq a_1 \leq a_2 \leq \cdots$ an increasing sequence of contractions in $A_+$, and $b := \sup_n a_n \in A^{**}$ its $\sigma(A^{**}, A^*)$-limit (i.e., weak limit). Denote by $e_{ij}$ the matrix units in $\mathcal{K}$.

**Lemma (13) (Dilation of increasing sequences)**

There is a unital *-morphism

$$h: \tilde{\mathcal{K}} := \mathcal{K} + \mathbb{C} \cdot 1 \to \mathcal{M}(\mathcal{K} \otimes A)$$

such that

$$e_{11} \otimes a_n = (e_{11} \otimes 1) h(p_n) (e_{11} \otimes 1),$$

with $p_n := e_{11} + e_{22} + \cdots + e_{nn}$ for all $n \in \mathbb{N}$. 
Idea of proof: Modify the Stinespring dilation of the unital c.p. map $V: \tilde{K} \to A$ with $V(e_{ij}) := (a_j - a_{j-1})\delta_{ij}$. Here $a_{-1} := 0$.

Remark (14)

Consider the hereditary C*-subalgebra

$$D := \bigcup_n h(p_n)(\mathcal{K} \otimes A)h(p_n) \subset \mathcal{K} \otimes A.$$

The open projection $p_D \in (\mathcal{K} \otimes A)^{\ast\ast} \cong \mathcal{L}(\ell_2) \overline{\otimes} A^{\ast\ast}$ corresponding to $D$ satisfies $(e_{11} \otimes 1)p_D(e_{11} \otimes 1) = e_{11} \otimes b$ and, for $w \in A \otimes B$:

$$e_{11} \otimes (w^*((1 - b) \otimes 1)w) = (e_{11} \otimes w)^*((1 - p_D) \otimes 1)(e_{11} \otimes w).$$

The last equation completes the proof of Proposition 9.
A noncommutative Egorov problem

Let \( A \) a unital or stable separable \( C^* \)-algebra, and let \( \mu \in A^* \) a positive linear functional on \( A \). The **central support** \( c \in \mathcal{Z}(A^{**}) \) of \( \mu \) is defined as the smallest projection \( c \) in the center \( \mathcal{Z}(A^{**}) \) of \( A^{**} \) with \( \mu(c) = \|\mu\| \). The usual support projection \( p_{\mu} \in A^{**} \) of \( \mu \) is not necessarily in the center of \( A^{**} \).

**Question (15) (nc Egorov)**

Let \( p \in A^{**} \) a projection, \( \varepsilon > 0 \).
Does there exists a **closed** projection \( q \in A^{**} \) such that

\[
qc \leq p \quad \text{and} \quad \mu(q) + \varepsilon > \mu(p).
\]

Recall here that a projection \( q \in A^{**} \) is **closed** if \( 1 - q \) is the **open** support projection \( p_D \) of a closed hereditary \( C^* \)-subalgebra \( D \subset A \).
If $A = C(\Omega)$ is commutative and unital, then the answer is positive and is equivalent to a theorem of Egorov in Measure theory.

There exists partial results that are related to a possible positive answer of question 15. But they are only generalization of a theorem of Lusin.

The above Lemma 13 (together with Remark 14) is a step towards a partial result, but only after stabilizing $A$ with the compact operators.
We obtain in a similar way a “stable” version of a non-commutative Egorov theorem (where we identify $e_{11} \otimes a$ with $a \in A^{**}$):

**Theorem (16) (nc Egorov)**

Let $A$ a separable unital C*-algebra and $\mu \in A^*$ a positive linear functional with central support (-projection) $c \in Z(A^{**})$, $T \in A^{**}$ with $\|T\| \leq 1$.

Then, for every $\varepsilon > 0$, there exists a **closed** projection $q \in (\mathcal{K} \otimes A)^{**}$ (i.e., $1 - q$ is the open support projection of an hereditary C*-subalgebra $D \subset \mathcal{K} \otimes A$) such that

$$(e_{11} \otimes c)q(e_{11} \otimes c) \leq T \quad \text{and} \quad \mu((e_{11} \otimes 1)q(e_{11} \otimes 1)) + \varepsilon > \mu(T).$$
What about a non-stable version of the characterization of local reflexivity, or only of $\| \cdot \|_C = \| \cdot \|$ on $A^{**} \otimes B$? (We may assume again that $A$ is unital and separable.)

The basic assumption is (equivalent to the assumption), that there is a universal constant $\rho < \infty$ with the property $\|(q \otimes 1)w\|_C \leq \rho \|(q \otimes 1)w\|$ for all closed projections $p \in A^{**}$ and all $w \in A \otimes B$.

Then we need as a property of $A$ that, for each countably decomposable projection $z \in \mathcal{Z}(A^{**})$ and each $b \in (A^1_+)_{\sigma_{\text{down}}}$, normal state $\mu \in A^{*} = (A^{**})_*$ on $A^{**}$ and every $\varepsilon > 0$, there is a closed projection $q \in A^{**}$ and an element $a \in A$ (both depending on $\varepsilon$) such that $za^*qa \leq \varepsilon 1 + (1 + \varepsilon) \sup_n b^{1/n}$ and $\mu(b) \leq 2\varepsilon + \mu(a^*qa)$.

The proof of the conclusion $\| \cdot \|_C = \| \cdot \|_{\text{min}}$ on $A^{**} \otimes B$ under this assumptions is similar to the above given proof of Proposition 9.
[AB] Archbold, R.J., Batty, J.K., $C^*-$tensor norms and slice maps, J. London Math. Soc. **22** [1980], 127–138. (Theorem 2.4.3)


Proof of “⇒” for Proposition 9:
If $\| \cdot \|_C = \| \cdot \|_{\min}$ on $A^{**} \otimes B$, then $\| \cdot \|_C = \| \cdot \|_{\min}$ on $(\mathcal{K} \otimes A)^{**} \otimes B$, by Lemma 4.

For closed left-ideals $L \subset \mathcal{K} \otimes A$ with open support projection $p_L \in (\mathcal{K} \otimes A)^{**}$ and $w \in (\mathcal{K} \odot A) \odot B$ holds
$$\text{dist}(w, L \odot B) = \|w((1 - p_L) \otimes 1)\|_C \quad \text{and} \quad \|w((1 - p_L) \otimes 1)\|_{\min} = \|(\pi_L \otimes \text{id}_B)(x)\|,$$ by Lemma 5.

Proof of Lemma 12:
We make repeated use of the non-commutative Lusin theorem (from M. Tomita in 1959, see book of G.K. Pedersen, Thm. 2.7.3): Given $x \in H$, $a_1, \ldots, a_n \in A^{**}$, $d_1 : A^{**} \to \mathcal{L}(H)$ normal *-representation $\varepsilon > 0$, then exist countably decomposable $p \in A^{**}$, $a'_k \in A$ with $a'_k p = a_k p$ and $\|x - px\| < \varepsilon$. Then $w = \sum_k a_k \otimes b_k \in A^{**} \odot B$ and $v := \sum_k a'_k \otimes b_k \in A \odot B$ satisfy $w(p \otimes 1) = v(p \otimes 1)$. 
The distance $|\rho(w^* w) - \rho((p \otimes 1)w^* w(p \otimes 1))|$ between $\|(d_1 \cdot d_2)(w)x\|^2$ and $\|(d_1 \cdot d_2)(w(p \otimes 1))x\|^2$ is $\leq \varphi(\varepsilon)$ for some increasing continuous function $\varphi$ with $\varphi(0) = 0$ (if given $x$ and $w$ are fixed).

$$\lambda((p \otimes 1)w^* w(p \otimes 1))^{1/2} \leq \|w(p \otimes 1)\|_C = \|v(p \otimes 1)\|_C = \|v(p \otimes 1)\|.$$ 

Since $\|v(p \otimes 1)\| = \|w(p \otimes 1)\| \leq \|w\|$ it implies

$$\lambda(w^* w) \leq f(\varepsilon) + \|w^* w\|_{\text{min}},$$

for each $\varepsilon > 0$, hence $\lambda(w^* w) \leq \|w^* w\|_{\text{min}}$. 

Now Lemma 2 says $\|w\|_C = \|w\|_{\text{min}}$. \hfill \square
More about the proof of Lemma 13: Consider the unital c.p. map $V: \mathcal{K} \to A$ with $V(e_{ij}) := (a_j - a_{j-1})\delta_{ij}$. Here $a_{-1} := 0$.

The Kasparov-Stinespring dilation defines a countably generated Hilbert $A$-module $H$ and a unital *-representation $k_1: \tilde{\mathcal{K}} \to \mathcal{L}(H)$ and a vector $x \in H$ such that $\langle x, k(c)x \rangle = V(c)$ for all $c \in \tilde{\mathcal{K}}$.

Then Kasparov triviality theorem gives an Hilbert $A$-module isomorphism $\gamma$ from $H \oplus H_A$ onto the Kasparov standard module $H_A$. Consider $H_A$ as the set $(\mathcal{K} \otimes A)(e_{11} \otimes 1)$ of first columns $\sum e_{n1} \otimes y_n$ in $\mathcal{K} \otimes A$. Then $\mathcal{L}(H_A)$ becomes naturally isomorphic with the multiplier algebra $\mathcal{M}(\mathcal{K} \otimes A)$ of $\mathcal{K} \otimes A$. 
Define \( k_2 : \tilde{\mathcal{K}} \to \mathcal{L}(H_A) \) by

\[
k_2(c) := \gamma(k_1(c) \oplus \chi(c)1)\gamma^{-1},
\]

where \( \chi : \tilde{\mathcal{K}} \to \mathbb{C} \) is the unique non-zero character.

The element \( w := \gamma(x) \) is a partial isometry in \( \mathcal{K} \otimes A \) with \( w^*w = e_{11} \otimes 1 \). Since \( A \) is unital, its stabilization \( \mathcal{K} \otimes A \) has stable rank one, and – therefore – we find a unitary

\[
U \in (\mathcal{K} \otimes A) + \mathbb{C} \cdot 1 \subset \mathcal{M}(\mathcal{K} \otimes A)
\]

with \( \gamma(x) = U(e_{11} \otimes 1) \).

The desired unital *-morphism \( h : \tilde{\mathcal{K}} \to \mathcal{M}(\mathcal{K} \otimes A) \) is given by

\[
h(c) := U^*k_2(c)U.
\]