Purely infinite C^*-algebras arising from crossed products

joint work with Mikael Rørdam

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Outline

Question

Which C-dynamical systems give raise to purely infinite crossed products?*

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- $a, b \in A^+, a \precsim b \iff \exists (x_n)$ in A such that $x_n^* b x_n \to a$.
- $a \in A^+$ is *properly infinite* if $a \oplus a \precsim a$.
- A is *purely infinite* if all positive elements are properly infinite.

We need to know something about the ideal structure of the crossed products!

Definition (S)

A C^*-dynamical system (A, G) is said to be *separating* if A separates ideals in $A \rtimes_r G$.

Proposition (S)

Let (A, G) be an C^*-dynamical system with G discrete. Then (A, G) is separating iff $a \precsim_* E(a) \precsim_* a$ for all positive elements in $A \rtimes_r G$.

- $a \precsim_* b \iff a \in \overline{AbA}$.
- $a \precsim b \implies a \precsim_* b$.
- $a \precsim b \iff a \precsim_* b$ if b is properly infinite.
- $a \precsim b \iff a \precsim_* b$ if A is abelian or purely infinite.

Proposition

Let (A, G) be a separating C^*-dynamical system with G discrete. Then $A \rtimes_r G$ is purely infinite iff

(i) *all $a \in A^+$ are properly infinite in $A \rtimes_r G$, and*

(ii) *$E(a) \precsim a$ for all $a \in (A \rtimes_r G)^+$.*

Proof: If $A \rtimes_r G$ is purely infinite (ii) follows from $E(a) \precsim_* a$. Conversely, if (i) and (ii) hold and $a \in (A \rtimes_r G)^+$, then

$$a \oplus a \precsim E(a) \oplus E(a) \precsim E(a) \precsim a,$$

which shows that a is properly infinite. □

Question

When is $a \precsim E(a)$, $a \in (A \rtimes_r G)^+$?

- (A, G) is *exact* = Every G-invariant ideal I in A induces an exact sequence

$$0 \longrightarrow I \rtimes_r G \xrightarrow{\iota \rtimes \mathrm{id}} A \rtimes_r G \xrightarrow{\pi \rtimes \mathrm{id}} A/I \rtimes_r G \longrightarrow 0$$

- (A, G) is exact iff $b \precsim_* E(b)$ for any $b \in (A \rtimes_r G)^+$, by (S)
- True if (A, G) is exact and $E(a)$ properly infinite
- True if $a \in C_c(G, A)$.
- False if (A, G) is not exact (how often does that happen?)

Proposition (S)

Let (A, G) be an exact C^*-dynamical system with G discrete such that the action of G on $\widehat{A}$ is essentially free. Then (A, G) is separating.

- G act *essentially free* on X = For every closed G-invariant subset $Y \subseteq X$, the set of points in Y with a trivial isotropy group, is dense in Y.
- $\widehat{A}$ = irreducible representations modulo unitary equivalence

Remark

- (A, G) separating $\Rightarrow$ (A, G) exact
- $(C(X), G)$ separating $\Rightarrow$ essentially free action, for G countable and $(C(X), G)$ amenable (i.e $C(X) \rtimes_r G$ is nuclear)

Question

When is $E(a) \precsim a$, $a \in (A \rtimes_r G)^+$?

Theorem

Let (A, G) *be an exact C*-dynamical system with* G *discrete. Suppose that the action of* G *on* $\widehat{A}$ *is essentially free, that* A *is separable, and that* A *has the ideal property (projections in* A *separate ideals). Then* $A \rtimes_r G$ *iff*

(i) *all* $a \in A^+$ *are properly infinite in* $A \rtimes_r G$.

Theorem

Let (A, G) be an exact C^*-dynamical system with G discrete and with $A = C_0(X)$ for some totally disconnected locally compact Hausdorff space X. Suppose that the action of G on X is essentially free. Then the following are equivalent:

- $A \rtimes_r G$ *is purely infinite.*
- *Every projection in* A *is properly infinite in* $A \rtimes_r G$.

Outline

Question

Which C-dynamical systems give raise to purely (or properly) infinite crossed products?*

Remark

- $C(X) \rtimes_r G$ is a purely (or properly) infinite crossed product $\Rightarrow G$ is non-amenable
- G is non-amenable $\Rightarrow C(X) \rtimes_r G$ is a properly infinite crossed product for some compact topological space X

Question

Does every non-amenable group G admit an action on some compact topological space X such that $C(X) \rtimes_r G$ is purely infinite?

Remark

$C(X) \rtimes_r G$ is a Kirchberg algebra $\Rightarrow$ G is countable, exact and non-amenable.

- countable, using separability
- exact, using nuclearity on $C^*_\lambda(G) \subseteq C(X) \rtimes_r G$
- non-amemable, using Følner sequesnces to produce a tracial state on $C(X) \rtimes_r G$

Theorem

Every countable, exact, and non-amenable group G admits an free, minimal, and amenable action on the Cantor set X such that $C(X) \rtimes_r G$ is a Kirchberg algebra in UCT class

One important step: If a direct sum $p \otimes \cdots \otimes p$ of n copies of a projection $p \in \ell^\infty(G)$ is properly infinite in $M_n(\ell^\infty(G) \rtimes_r G)$ then p itself is properly infinite in $\ell^\infty(G) \rtimes_r G$.

Theorem

Let G be a countable discrete group acting on a set X (eg. X could be G itself). The following are equivalent for every $E \subseteq X$:

(i) *The n-fold direct sum $p = 1_E \oplus \cdots \oplus 1_E$ is properly infinite in $M_n(\ell^\infty(X) \rtimes_r G)$ for some n.*

(ii) *There is no lower semicontinuous tracial weight τ on $\ell^\infty(X) \rtimes_r G$ for which $0 < \tau(1_E) < \infty$.*

(iii) *E is G-paradoxical.*

(iv) *1_E is properly infinite in $\ell^\infty(X) \rtimes_r G$.*

- τ tracial weight on A = $\mathbb{R}^+$-linear map $A^+ \to [0, \infty]$ with $\tau(a^*a) = \tau(aa^*), a \in A$
- E is G-paradoxical = There exist subsets $V_1, V_2, \ldots, V_{n+m}$ of E and $t_1, t_2, \ldots, t_{n+m} \in G$ such that both $(V_j)_1^n$ and $(V_j)_{n+1}^{n+m}$ cover E, but $(t_j.V_j)_1^{n+m}$ are pairwise disjoints subsets of E

Proof:

(i)$\Rightarrow$ (ii): Find projections e, f below p, equivalent to p. For a tracial weight τ we have that $\tau_n(p) \geq 2\tau_n(p)$ implying that $\tau_n(p)$ and $\tau(1_E)$ is either 0 or ∞.

$\neg$(iii)$\Rightarrow$ $\neg$(ii): Find a finitely additive G-invariant measure μ on $P(X)$ with $\mu(E) = 1$ (Tarski). Construct a lower semicontinuous G-invariant weight on $\ell^\infty(G)$ from μ and compose it with E.

(iii)$\Rightarrow$ (iv): Let $t \mapsto u_t$ be the unitary representation of G in $\ell^\infty(X) \rtimes_r G$, and put $s_1 = \sum_{j=1}^{n} u_{t_j} 1_{V_j}$ and $s_2 = \sum_{j=n+1}^{n+m} u_{t_j} 1_{V_j}$. It follows that $1_E = s_1^* s_1 = s_2^* s_2$ and $s_1 s_1^* + s_2 s_2^* \leq 1_E$. □

Corollary

Suppose that G is non-amenable. Then every projection $p \in \ell^\infty(G)$ which is full in $\ell^\infty(G) \rtimes_r G$ (not contained in a proper ideal in $\ell^\infty(G) \rtimes G_r$), is properly infinite in $\ell^\infty(G) \rtimes_r G$.

Proof: $1 \precsim p \oplus \cdots \oplus p \precsim 1 \oplus \cdots \oplus 1 \precsim 1.$

Remark

Hjorth and Molberg proved in 2006 that any countable discrete group G admits a free minimal action on the Cantor set X such that the crossed product C*-algebras has a tracial state.

Another important step: Find a suitable sub-C*-algebra A in $\ell^\infty(G)$ where every project $p \in A$ that is properly infinite in $\ell^\infty(G) \rtimes_r G$, is properly infinite $A \rtimes_r G$.

Proposition

Let G be countable exact group. There exist a separable G-invariant sub-C-algebra $C(X) = A$ in $\ell^\infty(X)$ such that*

(i) *The action of G on X is free and amenable ($A \rtimes_r G$ is nuclear)*

(ii) *The algebra A is generated by projections*

(iii) *Every project $p \in A$ that is properly infinite in $\ell^\infty(G) \rtimes_r G$, is properly infinite $A \rtimes_r G$*

Proof:

(i): For $t \neq e$ find partition $G = G_1 \dot{\cup} G_2 \dot{\cup} G_3$ st $G_j \cap tG_j = \emptyset$ for $j = 1, 2, 3$ ($\Rightarrow tx \neq x$ for $x \in \beta G$). Use G is exact iff $(\ell^\infty(G), G)$ is amenable iff exist a countable sequence in $C_c(G, \ell^\infty(G))$ st. .... Let N be the countable set ensuring the action free and amenable.

(ii) For $a \in \ell^\infty(G)$ find projections $(p_j)_1^{|\sigma(a)|}$ st. $C^*(a) = C^*(p_j)$.

(iii) Find countable subsets $N_0 = N, N_1, N_2, \ldots$ of $\ell^\infty(G)$ and countable G-invariant sets $P_0, P_1, P_2, \ldots$ of projections st $\mathcal{P}_{C^*(P_j)} \subseteq N_{j+1} \subseteq^{(ii)} C^*(P_{j+1})$ and $\mathcal{P}_{C^*(P_j)} \cap \mathcal{P}_{prop.inf.} \subseteq C^*(P_{j+1})$. Verify that $A = \overline{\cup_j C^*(P_j)}$ does the job. $\square$

Proof of Main Theorem:

Up on combining the two important steps we get a separable G-invariant sub-C*-algebra $C(X) = A$ in $\ell^\infty(G)$ such that

(i) The action of G on X is free and amenable ($A \rtimes_r G$ is nuclear)

(ii) The algebra A is generated by projections

(iii) Every project $p \in A$ that is properly infinite in $\ell^\infty(G) \rtimes_r G$, is properly infinite in $A \rtimes_r G$

(iv) Every project $p \in \ell^\infty(G)$ which is full in $\ell^\infty(G) \rtimes_r G$, is properly infinite in $\ell^\infty(G) \rtimes_r G$.

If the action is minimal: $0 \neq p \in A$ full in $A \rtimes_r G$, full in $\ell^\infty(G) \rtimes_r G$, properly infinite in $\ell^\infty(G) \rtimes_r G$, properly infinite $A \rtimes_r G$. In UCT do to Tu. □

Question

How do we compute K-theory?

Question

What is the range of this construction?

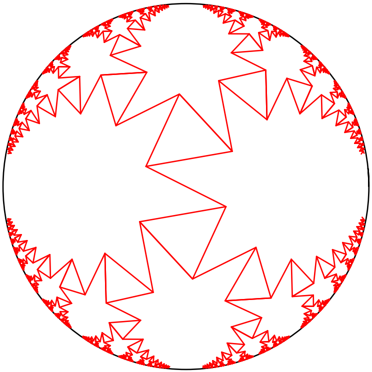

Figure: The Cayley graph for $\mathbb{Z}_2 * \mathbb{Z}_3$ with all edges of unit length.

Thank you for your attention :o)