C^*-algebras associated to dilation matrices

Iain Raeburn (University of Otago, NZ)

This talk is about joint work

with Ruy Exel and Astrid an Huef,
and with Marcelo Laca and Jacqui Ramagge

An *Exel system* (A, α, L) consists of an endomorphism α of a C^*-algebra A and a *transfer operator* L for α: a positive linear map $L : A \to A$ such that $L(\alpha(a)b) = aL(b)$ for $a, b \in A$.

Today A is unital, $\alpha(1) = 1$ and $L(1) = 1$. These can all be relaxed (look in the arXiv under “Brownlowe”).

Example. (*Classical Exel systems.*) Suppose that $\sigma : X \to X$ is a surjective local homeomorphism on a compact space X. Then $\alpha : f \mapsto f \circ \sigma$ is an endomorphism of $C(X)$, and there is a natural transfer operator:

$$L(f)(x) = \frac{1}{|\sigma^{-1}(x)|} \sum_{\sigma(y)=x} f(y).$$

Example. Fix $N \in \mathbb{N}$ and define $\alpha_N \in \operatorname{End} C(\mathbb{T})$ by $\alpha_N(f)(z) = f(z^N)$.

Example. The shift σ on the one-sided path space E^∞.

Suppose (A, α, L) is an Exel system. What is its crossed product? Exel (ETDS 03) advises as follows:

- View A as a right A-module A_L, by defining $m \cdot a = m\alpha(a)$ for $m \in A_L$ and $a \in A$.
- Define an A-valued pairing on A_L by $\langle m, n \rangle = L(m^* n)$; the properties of the transfer operator L ensure that A_L is then a pre-inner-product module over A.
- Complete to get a right Hilbert A-module M_L.
- Observe that $\phi(a) : m \to am$ extends to an adjointable operator on M_L, giving a homomorphism $\phi : A \to \mathcal{L}(M_L)$.
- Define $a \cdot m := \phi(a)m$ to make M_L into a right-Hilbert A–A bimodule (or correspondence over A, or Hilbert bimodule over A).

Now we simply define the *Exel crossed product* $A \rtimes_{\alpha,L} \mathbb{N}$ to be the Cuntz-Pimsner algebra $\mathcal{O}(M_L)$ of the bimodule.

Suppose that M is a right-Hilbert A–A bimodule. A *representation of M in a unital C^*-algebra B* consists of a linear map $\psi : M \to B$ and a homomorphism $\pi : A \to B$ such that

$$\psi(a \cdot m \cdot b) = \pi(a)\psi(m)\pi(b) \text{ and } \pi(\langle m, n\rangle) = \psi(m)^*\psi(n).$$

The *Toeplitz algebra $\mathcal{T}(M)$* is generated by a universal representation (i_M, i_A).

For $m, n \in M$, $\Theta_{m,n}$ denotes the rank-one operator $l \mapsto m \cdot \langle n, l\rangle$; $\mathcal{K}(M) := \overline{\text{span}}\{\Theta_{m,n} : m, n \in M\}$ is an ideal in $\mathcal{L}(M)$. A representation $(\psi, \pi) : M \to B$ induces a homomorphism $(\psi, \pi)^{(1)} : \mathcal{K}(M) \to B$ such that $(\psi, \pi)^{(1)}(\Theta_{m,n}) = \psi(m)\psi(n)^*$.

Say that (ψ, π) is *Cuntz-Pimsner covariant* if

$$\pi(a) = (\psi, \pi)^{(1)}(\phi(a)) \text{ whenever } \phi(a) \in \mathcal{K}(M).$$

The *Cuntz-Pimsner algebra $\mathcal{O}(M)$* is a quotient of $\mathcal{T}(M)$ which is generated by a universal C-P covariant rep (j_M, j_A).

C-P covariance often has very concrete implications. A finite sequence $\{m_i : 0 \leq i < N\}$ in M is a *Parseval frame* if

$$m = \sum_{i=0}^{N-1} m_i \cdot \langle m_i, m \rangle \text{ for every } m \in M.$$

One can interpret this *reconstruction formula* as saying that the identity 1 in $\mathcal{L}(M)$ is the finite-rank operator $\sum_{i=0}^{N-1} \Theta_{m_i,m_i}$. Thus C-P covariance of (ψ, π) implies that

$$\pi(1) = \sum_{i=0}^{N-1} (\psi, \pi)^{(1)}(\Theta_{m_i,m_i}) = \sum_{i=0}^{N-1} \psi(m_i)\psi(m_i)^*. \tag{1}$$

Indeed, since we also have $\phi(a) = \sum_{i=0}^{N-1} \Theta_{a \cdot m_i, m_i}$, C-P covariance is *equivalent* to (1)!

In (1), $\{\psi(m_i)\}$ is trying to be a Cuntz family. However, the $\psi(m_i)$ need not be isometries...

An *orthonormal basis* for M is a set $\{m_i : 0 \leq i < N\}$ which generates M and satisfies $\langle m_i, m_j \rangle = \delta_{i,j} 1$. Every orthonormal basis is a Parseval frame (just check the reconstruction formula on $m = \sum_i m_i \cdot a_i$ and extend by continuity to $m \in M$).

But the converse is not true: if $\{m_i\}$ is a Parseval frame, then $m \mapsto \{\langle m_i, m \rangle\}$ embeds M (as a right Hilbert module) onto a direct summand of A^N, and if P is the projection on M, then $m_i = Pe_i$ (Rieffel, Frank-Larson 03).

When $\{m_i\}$ is an orthonormal basis and (ψ, π) is a Cuntz-Pimsner covariant representation in B, we have

$$\psi(m_i)^* \psi(m_i) = \pi(\langle m_i, m_i \rangle) = \pi(1),$$

and $\{\psi(m_i)\}$ is a Cuntz family in $\pi(1) B \pi(1)$.

For $(C(\mathbb{T}), \alpha_N, L)$, the functions $\{z^i : 0 \le i < N\}$ form an orthonormal basis for M_L (Packer-Rieffel 03). Since $j_{C(\mathbb{T})}(1) = 1$, $\{j_{M_L}(z^i)\}$ is a Cuntz familiy. Other o/n bases are even more interesting.

Example. An orthonormal basis $\{m_i : 0 \le i < N\}$ for the right module M_L for $(C(\mathbb{T}), \alpha_N, L)$ is "a filter bank with perfect reconstruction". Of particular interest are filter banks where m_0 is "low-pass": $m_0(1) = N^{1/2}$, m_0 is smooth near 1, and $m_0 \neq 0$ on a sufficiently large neighbourhood of 1.

Consider the representation $M : C(\mathbb{T}) \to B(L^2(\mathbb{T}))$ and the isometry S on $L^2(\mathbb{T})$ given by $(S\xi)(z) = \xi(z^N)$, and define $\psi(f) = M(f)S$. Then (ψ, M) is a Cuntz-Pimsner covariant representation of M_L, and the corresponding Cuntz family $S_i := \psi(m_i)$ defined by $(S_i\xi)(z) = m_i(z)\xi(z^N)$ plays a fundamental role in the construction of wavelets (Bratteli-Jorgensen 97+, Larsen-R 06, Baggett et al 10).

All of which encouraged Astrid an Huef and I to have a closer look at the Exel crossed product $C(\mathbb{T}) \rtimes_{\alpha_N, L} \mathbb{N}$.

Results of Exel-Vershik show it is simple, and results of Katsura that it is purely infinite. So we computed its K-theory, finding

$$K_0 = \mathbb{Z} \oplus (\mathbb{Z}/(N-1)\mathbb{Z}) \text{ and } K_1 = \mathbb{Z}.$$

Then we started seeing this K-theory everywhere: this algebra has been studied in different guises by Deaconu, Brenken, Kajiwara and Watatani, Katsura, Yamashita, and....

There are other dilations of interest in wavelet theory. An integer matrix $A \in M_d(\mathbb{Z})$ is a *dilation matrix* if all its complex eigenvalues have $|\lambda| > 1$. It induces a covering σ_A of $\mathbb{T}^d$ such that $\sigma_A(e^{2\pi ix}) = e^{2\pi iAx}$, and an endomorphism α_A of $C(\mathbb{T}^d)$. If Σ is a set of coset representatives for $A^t\mathbb{Z}^d$ in $\mathbb{Z}^d$, then $\{z^m : m \in \Sigma\}$ is an orthonormal basis for M_L, and $\{j_{M_L}(z^m) : m \in \Sigma\}$ is a Cuntz family in $C(\mathbb{T}) \rtimes_{\alpha_A, L} \mathbb{N}$.

We proved (again leaning on Exel-Vershik and Katsura) that $C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N}$ is simple and purely infinite. Then we ran into a brick wall with the K-theory. Ruy Exel saved us by suggesting that, instead of trying to apply Pimsner's six-term sequence, we look at its proof and try to use the extra information we have, which is that M_L (as a rt module) has an orthonormal basis.

Suppose $(C(X), \alpha, L)$ is a classical Exel system and $\{m_i : 0 \leq i < N\}$ is an orthonormal basis for M_L.

Lemma. (EaHR) *Define* $\Omega : C(X) \to M_N(C(X))$ *by* $\Omega(f) = (\langle m_j, f \cdot m_k \rangle)_{j,k}$. *Then* Ω *is a homomorphism, and for* $f \in C(X)$, $\Omega(\alpha(f))$ *is the diagonal matrix* $f1_N$ *in* $M_N(C(X))$.

Notice that $K_i(M_N(C(X)))$ is canonically isomorphic to $K_i(C(X))$, and these isomorphisms (for both $i = 0$ and $i = 1$) take $[f1_N]$ to $N[f]$. So the lemma implies that $\Omega_* \circ \alpha_*$ is multiplication by N.

Theorem. (Exel–an Huef–R, 2010) *Let* $(C(X), \alpha, L)$ *be a classical Exel system, suppose that* $\{m_i : 0 \le i < N\}$ *is an orthonormal basis for* M_L*, and that* Ω *is as in the previous lemma. Then there is an exact sequence*

$$\begin{array}{ccccc} K_0(C(X)) & \xrightarrow{\mathrm{id} - \Omega_*} & K_0(C(X)) & \xrightarrow{j_{C(X)*}} & K_0(\mathcal{O}(M_L)) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}(M_L)) & \xleftarrow{j_{C(X)*}} & K_1(C(X)) & \xleftarrow{\mathrm{id} - \Omega_*} & K_1(C(X)). \end{array} \quad (2)$$

We want to apply this to $(C(\mathbb{T}^d), \alpha_A, L)$. Ji calculated

$$K_*(C(\mathbb{T}^d)) = K^*(\mathbb{T}^d) = H^*(\mathbb{T}^d, \mathbb{Z}) = \bigwedge H^1(\mathbb{T}^d, \mathbb{Z}) = \bigwedge [\mathbb{T}^d, \mathbb{T}].$$

$[\mathbb{T}^d, \mathbb{T}]$ is the free abelian group generated by the functions $u_k : z = (z_i, \cdots, z_d) \mapsto z_k$. The matrix of $(\alpha_A)_*$ wrt the basis $\{u_k\}$ is A^t. So $(\alpha_A)_* = \bigwedge A^t : \bigwedge \mathbb{Z}^k \to \bigwedge \mathbb{Z}^k$.

We want to compute Ω_* knowing that $\Omega_* \circ (\alpha_A)_*$ is multiplication by $N := |\det A|$ and $(\alpha_A)* = \bigwedge A^t : \bigwedge \mathbb{Z}^k \to \bigwedge \mathbb{Z}^k$. Over $\mathbb{Q}$, $\bigwedge A^t$ is invertible. So if we can find a matrix B over $\mathbb{Z}$ such that $B(\bigwedge A^t) = N 1_d$, then $\Omega_* = B$.

…a wee bit of multilinear algebra later, we have formulas for the kernel and cokernel of $\mathrm{id} - \Omega_*$, and the exact sequence on the previous slide gives short exact sequences for

$$K_0(\mathcal{O}(M_L)) = K_0(C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N}) \text{ and } K_1(\mathcal{O}(M_L)).$$

Examples. For $A = \left(\begin{smallmatrix} 2 & 1 \\ -1 & 2 \end{smallmatrix}\right)$, we have $\det A = 5 > 1$ and

$$K_0(\mathcal{O}(M_L)) = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z} \text{ and } K_1(\mathcal{O}(M_L)) = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}).$$

For $A = \left(\begin{smallmatrix} 2 & -1 \\ 1 & -3 \end{smallmatrix}\right)$, we have $\det A = -5$, and

$$K_0(\mathcal{O}(M_L)) = (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \text{ and } K_1(\mathcal{O}(M_L)) = \mathbb{Z}/5\mathbb{Z}.$$

An Exel crossed product $A \rtimes_{\alpha,L} \mathbb{N} = \mathcal{O}(M_L)$ has a dual action γ of $\mathbb{T}$ such that $\gamma_z(j_A(a)) = j_A(a)$ and $\gamma_z(j_{M_L}(m)) = zj_{M_L}(m)$, which lifts to $\sigma : t \mapsto \gamma_{e^{it}}$ of $\mathbb{R}$. What are the KMS states of $(C(\mathbb{T}^d) \rtimes_{\alpha_A,L} \mathbb{N}, \sigma)$?

In general, for $\beta \in [1, \infty)$, a **KMS$_\beta$ state** of (B, σ) is a state ψ such that $\psi(bc) = \psi(c\sigma_{i\beta}(b))$ for all b, c in a dense $*$-subalgebra of analytic elements (that is, elements such that $t \mapsto \sigma_t(b)$ is the restriction of an entire function).

In practice, it is easy to write down enough analytic elements. Here, $C(\mathbb{T}^d) \rtimes_{\alpha_A,L} \mathbb{N}$ is generated by a unitary representation $u : m \to j_{C(\mathbb{T}^d)}(z^m)$ of $\mathbb{Z}^d$ and an isometry $v = j_{M_L}(1)$. Then the elements $u_m v^k v^{*l} u_n^*$ span a dense $*$-subalgebra of $C(\mathbb{T}^d) \rtimes \mathbb{N})$, and are analytic: $\sigma_t(u_m v^k v^{*l} u_n^*) = e^{it(k-l)} u_m v^k v^{*l} u_n^*$.

The elements $\{u_m v = j_{M_L}(z^m) : m \in \Sigma\}$ are precisely the Cuntz family we had earlier, and hence any KMS$_\beta$ state ψ satisfies

$$\begin{aligned} 1 = \psi(1) = \psi\Big(\sum_{m\in\Sigma} u_m v v^* u_m^*\Big) &= \sum_{m\in\Sigma} \psi(v^* u_m^* \sigma_{i\beta}(u_m v)) \\ &= \sum_{m\in\Sigma} e^{i(i\beta)} \psi(v^* u_m^* u_m v) = e^{-\beta} \sum_{m\in\Sigma} \psi(1) \\ &= e^{-\beta} |\det A|. \end{aligned}$$

Theorem. (LRR) $(C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N}, \sigma)$ *has exactly one KMS state, and it occurs when* $\beta = \log|\det A|$.

Write $N := |\det A|$. To see there is a KMS$_{\log N}$ state, use the dual coaction of $\mathbb{Z}^d[A^{-1}] \rtimes \mathbb{Z}$ to build an expectation E onto $C := \overline{\operatorname{span}}\{u_m v^k v^{*k} u_m^*\}$. Then write $C = \overline{\bigcup_k C_k}$: each C_k is commutative with $\dim C_k = N^k$, so has a normalised trace, and these combine to give a trace τ on C. Take $\psi := \tau \circ E$.

We get a specific formula for $\psi(u_m v^k v^{*l} u_n^*)$, and playing with the KMS condition shows it is the only possibility.

Consider the Toeplitz algebra $\mathcal{T}(M_L)$ for $(C(\mathbb{T}^d), \alpha_A, L)$. It is also generated by a unitary representation u of $\mathbb{Z}^d$ and an isometry v, and the $u_m v^k v^{*l} u_n^*$ still span, but the Cuntz relation becomes an inequality. We still have a dynamics σ. Write $B = A^t$.

Theorem. (LRR) *For each $\beta > \log|\det A|$ and $\mu \in P(\mathbb{T}^d)$, there is a KMS$_\beta$ state $\psi_{\beta,\mu}$ such that $\psi_{\beta,\mu}(u_m v^k v^{*l} u_n^*)$ vanishes unless $k = l$ and $m - n \in B^k\mathbb{Z}^d$, and then equals*

$$(1-|\det A|e^{-\beta}) \sum_{\{j\geq k \,:\, n-m\in B^j\mathbb{Z}^d\}} |\det A|^{j-k} e^{-j\beta} \int_{\mathbb{T}^d} z^{B^{-j}(m-n)}\, d\mu(z).$$

Apart from the one lifted from the unique KMS state on $C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N}$, these are all the KMS states for $(\mathcal{T}(M_L), \sigma)$.

We build $\psi_{\beta,\mu}$ in the Hilbert space of the representation $\bigoplus_{j=0}^{\infty} \operatorname{Ind}_{B^j\mathbb{Z}^d}^{\mathbb{Z}^d} M \circ B^{-j}$, where M is the representation of $\mathbb{Z}^d$ on $L^2(\mathbb{T}^d, d\mu)$ defined by $(M_m f)(z) = z^m f(z)$.

References:

N. Brownlowe and I. Raeburn, Math. Proc. Camb. Phil. Soc. 141 (2006).

R. Exel, Ergodic Theory Dynam. Systems 23 (2003).

R. Exel, A. an Huef and I. Raeburn, Indiana Univ. Math. J., to appear (or see the arXiv).

Applications to wavelets:

N.S. Larsen and I. Raeburn, Proceedings of GPOTS 2005, Contemp. Math. 414 (2006).

L.W. Baggett, N.S. Larsen, J.A. Packer, I. Raeburn and A. Ramsay, J. Funct. Anal. 258 (2010).