A Remark on AH-algebras with diagonal maps

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AH-algebras

Definition
A C*-algebra A is called an AH-algebra if it has an inductive limit decomposition

$$A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow \varinjlim A_i \cong A$$

with $A_i \cong p_i \mathrm{M}_{n_i}(\mathrm{C}(X_i))p_i$ for a compact metrizable space X and a projection p_i.

Dimension growth

Consider an inductive system

$$A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow \varinjlim A_i$$

with $A_i \cong p_i \mathrm{M}_{n_i}(\mathrm{C}(X_i))p_i$. Write

$$p_i \mathrm{M}_{n_i}(\mathrm{C}(X_i))p_i = \bigoplus_{j=1}^{m_i} p_{i,j} \mathrm{M}_{n_{i,j}}(\mathrm{C}(X_{i,j}))p_{i,j}$$

such that $\mathrm{rank}(p_{i,j}(x))$ is constant on $X_{i,j}$. Then the *dimension growth* of the inductive system is

$$\liminf_{i\to\infty} \max\{\frac{\dim(X_{i,j})}{\mathrm{rank}(p_{i,j})};\ j = 1, ..., m_i\}.$$

The classification of AH-algebras

Definition

An AH-algebra has slow dimension growth if it has an inductive limit decomposition with dimension growth zero.

Theorem

The class of unital simple AH-algebra with slow dimension growth is classified by the Elliott invariant.

Theorem

There exists AH-algebra A (without slow dimension growth) such that

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$$

for some AI-algebra B but $A \ncong B$.

AH-algebra with diagonal maps

Let X and Y be two compact metrizable spaces. A unital homomorphism

$$\varphi : \mathrm{C}(X) \to \mathrm{M}_n(\mathrm{C}(Y))$$

is called a diagonal map if there are continuous maps

$$\lambda_1, ..., \lambda_n : Y \to X$$

such that

$$\varphi(f) = \begin{pmatrix} f \circ \lambda_1 & & \\ & \ddots & \\ & & f \circ \lambda_n \end{pmatrix}, \quad \forall f \in \mathrm{C}(X).$$

Definition

An AH-algebra A is called an AH-algebra with diagonal maps if it has an inductive limit decomposition

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \cdots \longrightarrow A = \varinjlim A_n$$

with $A_n = \mathrm{M}_{k_n}(\mathrm{C}(X_n))$ for some compact metrizable space X_n, and φ_n is diagonal.

Let α and β be open covers of X. Then,

1. $\beta \succ \alpha$ if for any $V \in \beta$, there is $U \in \alpha$ such that $V \subseteq U$.
2. $\alpha \vee \beta$ is the cover $\{U \cap V;\ U \in \alpha, V \in \beta\}$.
3. If $f : Y \to X$ is a continuous map, then

$$f^{-1}(\alpha) = \{f^{-1}(U);\ U \in \alpha\}$$

is an open cover of Y.

Let α be an open cover of a compact metrizable space X. Define

$$\mathrm{ord}(\alpha) = \max_{x \in X} \sum_{U \in \alpha} 1_U(x) - 1$$

and

$$\mathcal{D}(\alpha) = \min_{\beta \succ \alpha} \mathrm{ord}(\beta).$$

Example

If

$$\alpha = \{\underbrace{X, X, ..., X}_{n}\},$$

then $\mathrm{ord}(\alpha) = n - 1$, but $\mathcal{D}(\alpha) = 0$.

Remark

$$\mathcal{D}(\alpha) \leq \dim(X).$$

Consider the inductive system

$$\mathrm{M}_{n_1}(\mathrm{C}(X_1)) \xrightarrow{\varphi_1} \mathrm{M}_{n_2}(\mathrm{C}(X_2)) \xrightarrow{\varphi_2} \cdots$$

with

$$\varphi_k(f) = \begin{pmatrix} f \circ \lambda_{k,k+1}^{(1)} & & \\ & \ddots & \\ & & f \circ \lambda_{k,k+1}^{(m_k)} \end{pmatrix}.$$

Let α be a cover of X_1. On each X_k, it induces an open cover

$$\varphi_{1,k}(\alpha) := (\lambda_{1,k}^{(1)})^{-1}(\alpha) \vee \cdots \vee (\lambda_{1,k}^{(m)})^{-1}(\alpha),$$

where $\lambda_{1,k}^{(1)}, ..., \lambda_{1,k}^{(m)} : X_k \to X_1$ are eigenvalue functions of $\varphi_{1,k}$.

Compare $\mathcal{D}(\varphi_{1,k}(\alpha))$ with the matrix size n_k. The limit

$$r(\alpha) = \lim_{k \to \infty} \frac{\mathcal{D}(\varphi_{1,k}(\alpha))}{n_k}$$

exists if $n_k \to \infty$.

Definition (Lindenstrauss-Weiss)

The mean dimension of A is

$$\gamma(A) := \lim_{i \to \infty} \sup_{\alpha \text{ covers } X_i} r(\alpha).$$

Remark

AH-algebra with slow dimension growth always has mean dimension zero.

Theorem (Local Approximation)

Let A be an AH-algebra with diagonal maps, and denote by γ its mean dimension. Then, for any finite subset $\mathcal{F} \subset A$ and any $\varepsilon > 0$, there is a unital sub-C-algebra*

$$C \cong \mathrm{M}_n(\mathrm{C}(\Delta)) \subset A$$

such that $\mathcal{F} \subseteq_{\varepsilon} C$ and

$$\frac{\dim(\Delta)}{n} < \gamma + \varepsilon.$$

Corollary

Let A be an AH-algebra with diagonal maps. If A has mean dimension zero, then A has strict comparison on positive elements.

Corollary

Let A be an AH-algebra with diagonal maps. If A has mean dimension zero, then A is $\mathcal{Z}$-stable. Hence A is isomorphic to an AH-algebra without dimension growth.

Denote by ρ the restriction map

$$\rho : \mathrm{T}(A) \ni \tau \mapsto ([p] \mapsto \tau(p)) \in \mathrm{S}_u(\mathrm{K}_0(A)).$$

Then, one has the following remark on AH-algebra with diagonal maps.

Theorem

Let A be an AH-algebra with diagonal maps. If there exists $M > 0$ such that $\rho^{-1}(\kappa)$ has at most M extreme points for all $\kappa \in \mathrm{S}_u(K_0(A))$, or A has at most countably many extremal tracial states, then A has mean dimension zero, and hence is an AH-algebra without dimension growth.