C^*-algebras and finite approximation in dynamics

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Topological entropy

The entropy of a homeomorphism $T : X \to X$ of a compact metric space measures the exponential growth of the number of partial orbits up to an observational error. More precisely,

$$h_{\mathrm{top}}(T) = \sup_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{sep}(n, \varepsilon)$$

where $\mathrm{sep}(n, \varepsilon)$ is the maximal cardinality of an ε-separated set of partial orbits from 0 to $n - 1$.

Topological entropy

We can view a partial orbit from 0 to $n-1$ as a map

$$\varphi : \{0, \dots, n-1\} \to X$$

which is approximately equivariant with respect to the canonical cyclic permutation of $\{0, \dots, n-1\}$.

Notice that if k is proportionally small with respect to n then the map φ will be equivariant with respect to the actions of S^k and T^k on a proportionally large subset of $\{0, \dots, n-1\}$.

Topological entropy

Can one ever measure the entropy in a more strictly topological way by counting only those maps φ which are exactly equivariant, i.e., whose image is a periodic orbit? In several important cases yes, for example Bernoulli shifts $\mathbb{Z} \curvearrowright Y^{\mathbb{Z}}$. But there are positive entropy $\mathbb{Z}$-systems without periodic points, e.g., those which are minimal.

Topological entropy

We could nevertheless still retain the topological flavour of counting periodic points but weaken the requirement of exact equivariance by asking that φ be approximately equivariant on all of $\{0, \dots, n-1\}$ with respect to the actions of S^k and T^k for those values of k which are small relative to n, or even just $k = 1$. Then such a φ decomposes into approximate partial orbits whose length is small compared to n. By measuring the exponential growth with respect to n of the number of such φ to within an ε error one can recover the topological entropy in many cases, including that of minimal systems.

Actually, by merely asking that our maps φ be approximately equivariant on a large proportion of points in $\{0, \dots, n-1\}$ we always recover the topological entropy.

Topological entropy

Let G be a countable discrete group and let $G \curvearrowright X$ be a continuous action on a compact metrizable space. Let Σ be a sequence of maps $\sigma_i : G \to \mathrm{Sym}(m_i)$ into finite permutation groups which are asymptotically multiplicative and free. The existence of such a sequence defines a *sofic* group.

Let d be a compatible metric on X. On the set of maps $\{1, \ldots, m_i\} \to X$ define the pseudometric

$$\rho_2(\varphi, \psi) = \left(\frac{1}{m_i} \sum_{a=1}^{m_i} d(\varphi(a), \psi(a))^2 \right)^{1/2}.$$

Topological entropy

For a finite set $F \subseteq G$ and a $\delta > 0$ define $\mathrm{Map}(d, F, \delta, \sigma_i)$ to be the set of all maps $\{1, \dots, m_i\} \to X$ which are approximately F-equivariant to within δ. The image of such a map can be viewed as a system of interlocking partial orbits.

Set

$$h_\Sigma(d) = \sup_{\varepsilon>0} \inf_F \inf_{\delta>0} \limsup_{i\to\infty} \frac{1}{m_i} \log N_\varepsilon(\mathrm{Map}(d, F, \delta, \sigma_i))$$

where $N_\varepsilon(\cdot)$ denotes the maximal cardinality of an ε-separated set.

Proposition

$h_\Sigma(d)$ has a common value over all compatible metrics d.

Topological entropy

Definition

The topological entropy $h_\Sigma(X, G)$ is defined as the common value in the above proposition.

The entropy thus measures the exponential growth of the number of embedded sofic approximations up to an observational error.

One can similarly define sofic entropy for actions of G preserving a Borel probability measure μ, only now the maps from the sofic approximation into X must also approximately push forward the uniform probability measure to μ. In this case however it is much more difficult to show that one obtains a conjugacy invariant. In fact one must linearize the set-up and use operator algebra ideas.

Amenable case

In the case that G is amenable, Ornstein and Weiss's quasitiling theory shows that every sofic approximation $\sigma : G \to \mathrm{Sym}(d)$ approximately decomposes into Følner sets. Thus an approximately equivariant map $\{1, \dots, m\} \to X$ approximately decomposes into partial orbits over Følner sets.

Recall that the classical topological entropy $h_{\mathrm{top}}(X, G)$ is defined by measuring the exponential growth of the number of partial orbits with respect to an asymptotically invariant sequence of finite subsets of G, as in the integer action case from earlier.

Theorem

Suppose that G is amenable. Then $h_\Sigma(X, G) = h_{\mathrm{top}}(X, G)$.

As sofic approximations are by nature measure-theoretic objects, topological entropy as we have defined it is essentially a measure-theoretic quantity. Indeed we have the following simple relation between topological and measure entropy.

Theorem (variational principle)

Let $G \curvearrowright X$ be a continuous action of a countable sofic group on a compact metrizable space. Then

$$h_\Sigma(X, G) = \sup_\mu h_{\Sigma,\mu}(X, G)$$

where μ ranges over all invariant Borel probability measures on X.

Algebraic actions

Let f be an element in the group ring $\mathbb{Z}G$. Then G acts on $\mathbb{Z}G/\mathbb{Z}Gf$ by left translation, and this gives rise to an action α_f of G by automorphisms on the compact Abelian dual group $\widehat{\mathbb{Z}G/\mathbb{Z}Gf}$. This provides a rich class of actions which has been extensively studied in the case $G = \mathbb{Z}^d$ with connections to commutative algebra.

Recall that the Fuglede-Kadison determinant of an invertible element $a \in \mathcal{L}G$ is defined by $\det_{\mathcal{L}G} a = \exp \tau(\log |a|)$.

Theorem (Li)

Suppose that G is amenable and f is invertible in $\mathcal{L}G$. Then

$$h_{\text{top}}(\alpha_f) = \log \det_{\mathcal{L}G} f.$$

Algebraic actions

Suppose now that G is residually finite. Let $\{G_i\}_{i=1}$ be a sequence of finite-index normal subgroups of G with $\bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} G_i = \{e\}$, and let $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(G/G_i)\}_{i=1}^{\infty}$ be the associated sofic approximation sequence.

Theorem

Suppose that f is invertible in $C^*(G)$. Then

$$h_\Sigma(\alpha_f) = \log \det_{\mathcal{L}G} f.$$

The entropy in this case is equal to the exponential growth rate of the number of G_i-fixed points.

The existence of embedded sofic approximations for a measure-preserving action $G \curvearrowright (X, \mu)$ implies that $L^\infty(X, \mu) \rtimes G$ embeds into R^ω, i.e., admits tracial microstates. To formulate a topological analogue of this for a continuous action $G \curvearrowright X$ we should instead consider approximately equivariant maps into X from a finite set on which G acts genuinely.

Definition

We say that the action $G \curvearrowright X$ is *residually finite* if or every finite set $F \subseteq G$ and $\varepsilon > 0$ there are a finite set E, an action of G on E, and a map $\zeta : E \to X$ such that $\zeta(E)$ is ε-dense in X and $d(\zeta(sz), s\zeta(z)) < \varepsilon$ for all $z \in E$ and $s \in F$.

Note that G admits a residually finite free action if and only if it is a residually finite group.

One can formulate a noncommutative analogue of a residually finite action as follows.

Definition

We say that an action α of G on a separable unital C^*-algebra A is *quasidiagonal* if there exist unital completely positive maps $\varphi_n : A \to M_{k_n}$ and actions $G \curvearrowright M_{k_n}$ such that

(1) $\lim_{n\to\infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0$ for all $a, b \in A$,

(2) $\lim_{n\to\infty} \|\varphi_n(a)\| = \|a\|$ for all $a \in A$, and

(3) $\lim_{n\to\infty} \|\varphi_n(s \cdot a) - s \cdot \varphi_n(a)\| = 0$ for all $a \in A$ and $s \in G$.

Note that residually finite actions are quasidiagonal.

Can one use the existence of maps satisfying the above conditions, or a 2-norm version of them, to define a notion of entropy for actions of sofic groups on C^*-algebras?

Recall that a separable C^*-algebra A is an *MF algebra* if it can be embedded into $\prod_{n=1}^{\infty} M_{k_n} / \bigoplus_{n=1}^{\infty} M_{k_n}$ for some sequence $\{k_n\}_{n=1}^{\infty}$. This is equivalent to the existence of norm microstates, or to the existence of *-linear maps $\varphi_n : A \to M_{k_n}$ such that

(1) $\lim_{n\to\infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0$ for all $a, b \in A$, and

(2) $\lim_{n\to\infty} \|\varphi_n(a)\| = \|a\|$ for all $a \in A$.

Theorem

Suppose that $C^*_\lambda(G)$ is an MF algebra. Let α be a quasidiagonal action of G on a separable C^*-algebra A. Then $A \rtimes_\lambda G$ is an MF algebra. Moreover, if $C^*_\lambda(G)$ is quasidiagonal and A is nuclear then $A \rtimes_\lambda G$ is quasidiagonal.

Haagerup and Thorbjørnsen showed that $C^*_\lambda(F_r)$ is an MF algebra for all $r \geq 1$.

Theorem (Pimsner)

For a metrizable $\mathbb{Z}$-system (X, T) the following are equivalent:

1. (X, T) is residually finite (i.e., chain recurrent),
2. $C(X) \rtimes_\lambda \mathbb{Z}$ can be embedded into an AF algebra,
3. $C(X) \rtimes_\lambda \mathbb{Z}$ is quasidiagonal,
4. $C(X) \rtimes_\lambda \mathbb{Z}$ is stably finite.

To prove (4) $\Rightarrow$ (1) above one uses the fact that the lack of residual finiteness is equivalent to the compressibility of some open set, which yields a nonunitary isometry in $C(X) \rtimes_\lambda \mathbb{Z}$. This characterization of residual finiteness in terms of incompressibility is unavailable for actions of other groups.

Nevertheless, by directly extracting finite approximations from the existence of norm microstates we can show the following.

Theorem

Let $r \geq 2$. Let $F_r \curvearrowright X$ be a continuous action on the Cantor set. Then the action is residually finite if and only if $C(X) \rtimes_\lambda F_r$ is an MF algebra.

For every nonamenable G one can construct a continuous action of G which is not residually finite although all of its restrictions to cyclic subgroups are residually finite:

Take an action $G \curvearrowright X$ that does not admit an invariant probability measure. Then the induced action $G \curvearrowright M_X$ on the space of probability measures is not residually finite, as this would give the existence of an invariant measure on M_X, and hence also on X by taking the barycentre. But the restriction of the action to any cyclic subgroup H is residually finite, as one can show using the affine structure of M_X and the existence of an H-fixed point.

It follows that some matrix algebra over $C(X) \rtimes_\lambda G$ contains a nonunitary isometry. However one cannot construct such an isometry by using the compression of an open set by a single group element as in the case of $\mathbb{Z}$.

Theorem

Let $G \curvearrowright X$ be a continuous action on a compact metrizable space. Then the following are equivalent:

1. X is completely $(G, \mathscr{B}_X)$-nonparadoxical,
2. X is countably $(G, \mathscr{B}_X)$-nonparadoxical,
3. there is a G-invariant Borel probability measure on X.

The equivalence of (2) and (3) is a special case of a result of Becker and Kechris which builds on work on Nadkarni.

Theorem

Let $r \geq 1$ and let $F_r \curvearrowright X$ be a minimal continuous action on a compact metrizable space. Then the following are equivalent:

1. the action is residually finite,
2. there is an F_r-invariant Borel probability measure on X,
3. $C(X) \rtimes_\lambda F_r$ is an MF algebra,
4. $C(X) \rtimes_\lambda F_r$ is stably finite,
5. X is completely $(F_r, \mathscr{B}_X)$-nonparadoxical,
6. X is countably $(F_r, \mathscr{B}_X)$-nonparadoxical.

If X is zero-dimensional then we can add the following to the list:

7. every nonempty clopen subset of X is completely $(F_r, \mathscr{C}_X)$-nonparadoxical.