ASH-inductive limit: Approximation by Elliott-Thomsen building blocks

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(The talk was presented by using black board. The following is the material written on the blackboard as I can recall)

Recursive sub-homogeneous algebras(r.s.h. algebras) (Phillips): Unital C^*-algebras defined recursively:

1. M_n is r.s.h. $\forall n \in \mathbb{N}$.

2. If B is r.s.h., X is compact metric space, $X^0 \subset X$ is a closed set, $\phi : \ B \to M_n(C(X))$ is a unital homomorphism, then

$$A = B \oplus_{(M_n(C(X^0)),\phi)} M_n(C(X))$$

$$= \{(b, f) \in B \oplus M_n(C(X)),\ \phi(b) = f|_{X^0}\}$$

is a r.s.h. algebra.

$$\begin{array}{ccc} A & \longrightarrow & M_n(C(X)) \\ \downarrow & & \downarrow r \\ B & \longrightarrow & M_n(C(X^0)) \end{array}$$

where r is restriction.

ASH-inductive limit: A is unital inductive limit

$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \cdots$$

with each A_n r.s.h.

AH inductive limit:

$$A_n = P_n M_{[n]}(C(X_n))P_n,$$

where X_n are compact metric spaces, and $P_n \in M_{[n]}(C(X_n))$ are projections.

Rordam (2003): Constructed counterexample for Elliott conjecture.

Toms (2005): Such counterexample exists in the class of simple AH algebras (so we need slow dimension growth in the following theorem).

Theorem (Elliott-Gong-Li, Winter-Lin) if both A and B are simple AH algebras with (very) slow dimension growth, then $A \cong B \Leftrightarrow Ell(A) \cong Ell(B)$.

$Ell(A) = (K_0(A), K_0(A)^+, [\mathbf{1}_A], K_1(A), TA, < \ , \ >)$

Theorem (Elliott-Gong, Dadarlat-Gong) For the case that both A and B are simple AH algebras of real rank zero with slow dimension growth, $A \cong B$ if and only if

$$(K_0(A), K_0(A)^+, [\mathbf{1}_A], K_1(A)) \cong (K_0(B), K_0(B)^+, [\mathbf{1}_B], K_1(B))$$

Decomposition Theorem — one of the most important ingredients.

Elliott-Gong: real rank zero AH algebra

Gong: simple AH algebras.

Huaxin Lin gives an abstraction of the decomposition theorem. He calls it TA(something)

<u>Definition</u>. (Lin) Let $\mathscr{S}$ be a class of "good" unital C^*-algebras. A simple C^*-algebra A is called $TA\mathscr{S}$ if for any finite set $F \subset A,\ \varepsilon > 0,\ a \in A_+$, there exists a $B \in \mathscr{S}$ such that (denote $p = \mathbf{1}_B \in A$)

(1) $\|xp - px\| < \varepsilon \qquad \forall x \in F$;

(2) $d(pxp, B) < \varepsilon \qquad \forall x \in F$;

(3) $1 - p$ is equivalent to a projection in $Her(a)$.

Lin studied the following two cases.

1. If $\mathscr{S} = \{\oplus_i M_{k_i}\}$, then $TA\mathscr{S} \triangleq TAF$.
2. If $\mathscr{S}=\{\oplus_i M_{k_i} \oplus \oplus_j M_{l_j}(C[0,1])\}$, then $TA\mathscr{S} \triangleq TAI$.

Theorem (a) (Elliott-Gong) All real rank zero AH algebras (with slow dimension growth) are TAF.

(b) (Gong) All simple AH algebras (with very slow dimension growth) are TAI.

(We proved this theorem before Lin introduced TAF and TAI.)

Remarkably, Lin classified all simple TAF, TAI with UCT (without assume inductive limit structure).

Niu classified simple TAS with UCT, where S={splitting interval algebras}.

But many unital ASH algebras have no nontrivial projections. So it will not be $TA\mathscr{S}$ for any "good" class $\mathscr{S}$.

Winter makes important break through.

Lin-Niu: If $A \otimes UHF$ is TAF, then $A \otimes Z$ can be classified by Elliott invariant (a important special case was due to Winter), where Z is the Jiang-Su algebra.

<u>Definition</u> A is called rationally $TA\mathscr{S}$ if $A \otimes UHF_{\mathbb{Q}}$ is $TA\mathscr{S}$.

Theorem. (Gong) All simple AH algebras are rationally TAI.

Theorem. (Lin) If A is rationally TAI, then $A \otimes Z$ is classified by Elliott invariant.

Theorem. (Winter,) All simple AH algebras with slow dimension growth are Z-stable: $A \cong A \otimes Z$. (using result of Brown-Perera-Toms)

(Winter: For C^*-algebra A with locally finite nuclear dimension $A \cong A \otimes Z$ if and only if $Cu(A) \cong Cu(A \otimes Z)$.

Brown-Perera-Toms: If A is a simple AH algebra with slow dimension growth., then Cunts semigroup can be recovered from Elliott invariant, in particular, $Cu(A) \cong Cu(A \otimes Z)$.)

Combining above three theorems, we get classification of simple AH algebras with slow dimension growth.

With remarkable work of Lin and Winter, my decomposition theorem is still a necessary step.

But rationally TAI or rationally TAS do not include all simple AH algebras. So we need a new class of C*-algebras to replace F, I or S. The class can not be too small (otherwise rationally TA(new class)can not include all simple ASH algebras) and also can not be too large (otherwise it will not be useful for the classification).

Class $\mathcal{ET}$ (Elliott-Thomsen building blocks)

Let F_1 and F_2 be finite dimensional and $\alpha_0, \alpha_1 : F_1 \to F_2$ be unital homomorphisms.

$$A = \{(a, f) \in F_1 \oplus C([0,1], F_2) | f(0) = \alpha_0(a), f(1) = \alpha_1(a)\}$$

is called an Elliott-Thomsen building block, denoted by $A \in \mathcal{ET}$. If $K_1(A)$ is finite, then $A \in \mathcal{ET}_0$, our $\mathscr{S}$ will be $\mathcal{ET}_0$.

(I would like to mention two important theorems at early stage of classification of simple AH algebras.)

Theorem 1. (Elliott 1993) If both A and B are simple AI algebras, then $A \cong B \Longleftrightarrow Ell(A) \cong Ell(B)$.

Theorem 2. (Li 1999, Crelle's Journal) If $A = \lim(A_n, \phi_{nm})$ is a simple AH inductive limit with slow dimension growth, $\forall F \subset A_n$ finite, $\varepsilon > 0, \exists m, B = \oplus M_{k_i}(\mathbb{C}) \oplus \oplus M_{l_i}(C[0,1])$ and a homomorphism $\psi = A_n \to B \to A_m$ such that for any $\tau \in TA_m$,

$$\|\tau(\phi_{nm}(f) - \tau(\psi(f)\| < \varepsilon \qquad \forall f \in F.$$

Li's theorem make it possible to prove Gong's decomposition theorem: Simple AH algebras (with very slow dimension growth) are TAI.

Lin and Winter's work really reduced the case of simple TAI(rationally TAI) to simple AI algebras of Elliott.

The following are main theorems of this talk:

Theorem A. Let $A = \lim(A_n, \phi_{nm})$ be simple ASH-inductive limit (with slow dimension growth) of r.s.h. algebras. Then for any finite set $F \subset A$ and $\varepsilon > 0$, there exists a $B \in \mathcal{ET}_0$ and a homomorphism $\psi : A_n \to B \to A_m$ such that $\forall \tau \in TA_m$,

$$\|\tau(\phi_{nm}(f) - \tau(\psi(f)\| < \varepsilon \qquad \forall f \in F.$$

Theorem B. If $A = \lim(A_n, \phi_{nm})$ and $B = \lim(B_n, \psi_{nm})$ are simple unital inductive limit with $A_n, B_n \in \mathcal{ET}_0$, then $A \cong B \Longleftrightarrow Ell(A) \cong Ell(B)$.

Robert also (independently) obtained a result which covered the special case of our theorem for $K_1(A_n) = K_1(B_n) = 0$

Future goals:

(1) Classify all simple Z-stable rationally TA$\mathcal{ET}_0$ class algebras. Theorem B should play important role, like the role of Elliott classification theorem for simple AI algebras(see Theorem 1 above) in classification rationally TAI algebras.

(2) Prove that all unital simple ASH algebras are rationally TA$\mathcal{ET}_0$. Theorem A should play important role, like the role of Li's theorem on Crelle's Journal(see Theorem 2 above) in the proof of Gong's decomposition theorem.

Once these two goals are realized, then the complete classification of simple ASH algebras with slow dimension growth is obtained (combining theorems of Winter and Toms, we know all simple ASH algebras with slow dimension growth are Z-stable).