SOME GENERALIZATIONS OF VOICULESCU'S NON-COMMUTATIVE WEYL-VON NEUMANN THEOREM.

JOINT WORK WITH ALIN CIUPERCA,
ZHUANG NIU
PING NG.

THM (VOICULESCU, 1976).

A UNITAL SEP. C^*-ALG.

H SEP. INFINITE DIMEN. HILBERT SPACE

$\varphi, \psi : A \longrightarrow B(H)$ UNITAL $*$-HOMOM.

ST. $\varphi(A) \cap \mathcal{K} = \psi(A) \cap \mathcal{K} = \{0\}$

THEN φ AND ψ APPROX. UNITARILY EQUIVALENT

RECALL: $\varphi \sim_a \psi$ IF $\exists (u_n)_{n \geq 1}$, $u_n \in \mathcal{U}(H)$

st. $\forall a \in A$

$$\| \varphi(a) - u_n \psi(a) u_n^* \| \longrightarrow 0$$

1981, HADWIN

THM: A UNITAL SEP. C^*-ALG.

$\varphi, \psi : A \longrightarrow B(H)$ $*$-HOMOM.

THEN, TFAE:

① $\varphi \sim_K \psi$ (APPROX. UNIT. EQUIV. UP TO COMP.)

(I.E. $\varphi \sim_a \psi$ AND $\forall a \in A$, $\forall u \geq 1$

$(\varphi(a) - u_u \psi(a) u_u^*) \in \mathcal{K}$)

② $\varphi \sim_a \psi$

③ $\varphi \sim_{wa} \psi$ (WEAK APPROX. UNIT. EQUIV)

(I.E. $\exists\, (u_u)_{u \geq 1}, (J_u)_{u \geq 1}$ SEQ. IN $\mathcal{U}(H)$ S.T.

$\forall a \in A$, $\varphi(a) - u_n \psi(a) u_u^* \xrightarrow[\text{WOT}]{} 0$

$\psi(a) - J_u \varphi(a) J_u^* \xrightarrow[\text{WOT}]{} 0$)

④ RANK $(\varphi(a))$ = RANK $(\psi(a))$, $\forall a \in A$.

<u>GOAL</u>: LOOK AT "LOCALIZED" VERSIONS OF <u>HADWIN – VOICULESCU THM.</u>

TWO DIRECTIONS :

(1) SPECIALIZE THE TARGET ALG.

* REPLACE $B(H)$ BY OTHER VN. FACTORS.
* REPLACE POINTWISE – NORM CONV BY POINTWISE – $\sigma(M, M_*)$ – CONV. (M. VN FACTOR).

(2) SPECIALIZE THE CLASS OF C^*-ALGEBRAS.

* TAKE A NUCLEAR.
* REPLACE $B(H)$ BY A GENERAL VN ALG. M

  REPLACE POINTWISE - NORM CONV BY POINTWISE - $\sigma(M, M_*)$ - CONV.

SOME KNOWN RESULTS.

(2005) DING-HADWIN: $\mathcal{R}$ vNa. $x \in \mathcal{R}$.

$$\mathcal{R}\text{-RANK}(x) = \text{M-vN EQUIV CLASS OF PROJ. ON } \overline{\text{RANGE}(x)}.$$

* A UNITAL C^*-ALG. $\pi, \rho : A \to \mathcal{R}$ UNITAL.

$$\pi \sim_a \rho \ (\mathcal{R}) \Rightarrow \mathcal{R}\text{-RANK} \circ \pi = \mathcal{R}\text{-RANK} \circ \rho$$

* A UNITAL C^*-ALG., $\mathcal{M}$ FINITE FACTOR.
τ F.N TRACE ON $\mathcal{M}$.

$$\mathcal{M}\text{-Rk} \circ \pi = \mathcal{M}\text{-Rk} \circ \rho \quad \text{IFF} \quad \tau \circ \pi = \tau \circ \rho$$

* A SEP. UNITAL AH-ALG.
$\mathcal{M} \subset B(H)$, H SEP. $\mathcal{M}$ FINITE FACTOR

$$\pi \sim_a \rho \quad \text{IFF} \quad \mathcal{M}\text{-Rk} \circ \pi = \mathcal{M}\text{-Rk} \circ \rho$$

IFF $\hat{\tau} \circ \pi = \tau \circ \rho$, FOR τ NORMALIZED FAITHFUL, NORMAL TRACE τ ON $\mathcal{M}$.

HADWIN SHERMAN $\exists$ FINITE VN FACTOR M

SEP. UNITAL C^*-ALG A

$\varphi, \psi : A \longrightarrow M$ UNITAL, $*$-HOM. ST. $\tau \circ \varphi = \tau \circ \psi$

BUT $\varphi \not\sim_a \psi$.

EVEN $\varphi \not\sim_{wa} \psi$

COME BACK TO DIRECTION (1).

* REPLACE $B(H)$ BY

M PROPERLY INF. V.N FACTOR WITH M_* SEP.

M SEMIDISCRETE

(ie. id_M FACTORS TROUGH MATRIX-ALG.

IN THE PT-WISE - $\sigma(M, M_*)$ - TOPOL.).

RECALL: M VN FACTOR, M_* SEP.

- SEMIDISCRETE $\Longleftrightarrow$ INJECTIVE
- INJECTIVE $\Longleftrightarrow$ AFD.

CONNES - HAAGERUP :

$\exists !$ AFD FACTOR OF TYPE II_1, II_∞, III_λ, $\lambda \neq 0$

$$\Big\{ \text{TYPE } \mathrm{III}_0 \text{ AFD FACTORS} \Big\} \Big/ \text{ISOM} \longleftrightarrow \Big\{ \text{NON TRANSITIVE ERG. FLOW} \Big\} \Big/ \text{CONJUG}$$

THM: M TYPE III SEPARABLE SEMIDISCRETE VN FACTOR

A SEP. UNITAL C^*-ALG.

$\varphi, \psi : A \to M$ INJECTIVE UNITAL $*$-HOM.

THEN $\exists\ (u_n)_{n \geq 1}$ SEQ. OF UNITARIES IN M ST.

$\forall a \in A \quad \left(u_n \varphi(a) u_n^* - \psi(a)\right) \longrightarrow 0 \quad s^*(M, R_\varepsilon)$

COR: M, A AS ABOVE

$\varphi, \psi : A \to M$ UNITAL $*$-HOMOM.

THEN, TFAE:

① $\ker \varphi = \ker \psi$

② φ AND ψ ARE STRONG*- APPROX. UNIT. EQUIV.

③ φ AND ψ ARE WEAK*- APPROX. UNIT. EQUIV.

FOR A TYPE II_∞ FACTOR M, SITUATION MORE COMPLICATED DUE TO $\mathcal{K}_M$, THE IDEAL OF COMPACT OP. OF M.

RECALL: $\mathcal{K}_M$ NORM CLOSURE OF $F_M = \{ m \in M ;$

$\exists\, p$ proj of M ST. $\tau(p) < \infty$ AND $m = pmp \}$.

THM: M TYPE II_∞ SEMIDISCRETE FACTOR

τ F.N.SF. TRACE ON M (UNIQUE UP TO CONST.)

A UNITAL SEP. C^*-ALG.

$\varphi, \psi : A \longrightarrow M$ UNITAL INJECTIVE $*$-HOMO.

ST. $\tilde{\tau} \circ \varphi = \hat{\tau} \circ \psi$

THEN φ AND ψ ARE STRONG*- APPROX. UNIT. EQUIV.

TOWARDS A CONVERSE STATEMENT:

DEF: C, D C^*-ALG. $\varphi : C \to D$ $*$-HOMO.

φ IS **FULL** IF $\forall a \neq 0$, $a \in C^+$, $\varphi(a)$ FULL ELT OF D

(i.e. $\varphi(a)$ NOT CONTAINED IN ANY PROPER C^*-IDEAL OF D)

REM: 1) φ FULL $\Rightarrow$ φ INJECTIVE.

2) $\left.\begin{array}{l}\varphi : A \to M, \ M \text{ TYPE III} \\ \ker \varphi = \{0\}\end{array}\right\} \Longrightarrow \varphi$ FULL

M SIMPLE !

3) $\left.\begin{array}{l}\varphi : A \to M \text{ FULL} \\ M \text{ TYPE } \mathrm{II}_\infty\end{array}\right\} \Rightarrow \forall a \in A_+, \ a \neq 0$

$\tau(\varphi(a)) = \infty$.

IN PARTICULAR, IF $\varphi, \psi : A \to M$ UNITAL, FULL $*$-HOMO.

THEN $\tau \circ \varphi = \tau \circ \psi$

NEXT STEP: $\psi = \eta \circ \sigma$ CAN BE APPROX. IN PT-WISE NORM TOPOLOGY BY MAPS OF THE FORM $a \mapsto v^* a v$, v ISOMETRY OF M.

HENCE, $\forall a \in F$, $\| \varphi(a) - v^* a v \|^{\#}_{\chi}$ SMALL.

M PROPERLY INFINITE

$\underset{\text{HAAGERUP}}{\Longrightarrow}$ $\exists\, u$, UNITARY, ST $\| \varphi(a) - u^* a u \|^{\#}_{\chi}$ SMALL.

THM; M TYPE III FACTOR TFAE

① M INJECTIVE

② A UNITAL SEP. C^*-ALG.

$\varphi, \psi : A \to M$ INJECTIVE, UNITAL $*$-HOM.

THEN φ AND ψ $\sigma(M, M_*)^*$- APPROX. UNIT. EQU

SKETCH OF THE PF: $\chi \in M_*^+$, χ FAITHFUL

ENOUGH TO SHOW: $x_1, -, x_n \in M$ GIVEN

$\exists$ F FIN. DIMEN. FACTOR OF M; $y_1, -, y_n \in F$

ST $x_i \sim y_i$ IN $\|-\|_\chi^\#$ - NORM.

$A = C^*(x_1, -, x_n)$, $M = N \otimes B(H)$.

$\exists$ $\psi : A \longrightarrow 1 \otimes B(H) \subset N \otimes B(H) = M$ INJECT. UNITAL

BY HYP. $\exists$ $u \in M$, UNITARY ST

$x_e \sim u\psi(x_e)u^*$ IN $\|-\|_\chi^\#$ - NORM.

$B(H)$ INJECTIVE: $\exists$ FIN. DIM. FACTOR $F' \subset 1 \otimes B(H)$

$b_1, -, b_n \in F'$

$\psi(x_e) \sim b_e$ IN $\|-\|_{\mathrm{Ad}u \circ \chi}^\#$ -NORM

SET $F = uF'u^*$, $y_e = u b_e u^*$.

THM: M PROPERLY INFINITE VN FACTOR, M_* SEP.

TFAE ① M SEMI-DISCRETE

② A UNITAL SEP. C^*-ALG.

$\varphi, \psi : A \longrightarrow M$ UNITAL, FULL $*$-HOMO.

THEN φ AND ψ ARE σ-STRONG* APPROX. UNITARILY EQUIV.

REM: ① II_1 CASE: M SEMIDISCRETE. VN FACTOR.

$\varphi, \psi : A \to M$ UNITAL $*$-HOMO ST

$$\hat{\tau} \circ \varphi = \hat{\tau} \circ \psi$$

THEN φ AND ψ ARE σ-STRONG* APPROX. UNIT. EQUIV.

② CONVERSE OPEN.

③ [$\forall A$ UNITAL, SEP., $\forall \varphi, \psi : A \to M \otimes M$ ST $\hat{\tau} \circ \varphi = \hat{\tau} \circ \psi$. THEN φ AND ψ σ-STRONG* APPROX. UNIT. EQUIV.]

$\Longrightarrow$ M SEMIDISCRETE

THM: M TYPE III INJECTIVE FACTOR, M_* SEP.

A SEP. UNITAL SUB-C^*-ALG. OF M.

$\varphi: A \to M$ INJECTIVE $*$-HOMO.

THEN, $\exists\ (u_n)_{n\geq}$ SEQ OF UNITARIES OF M. ST.

$$\forall a \in A,\quad \left(\varphi(a) - u_n a u_n^*\right) \xrightarrow[s(M, M_*)]{} 0$$

SKETCH OF THE PF: $\chi \in M_*^+$, FAITHFUL.

$\varepsilon > 0$, $\mathcal{F} \subset A$, FINITE.. CAN ASSUME: $\mathcal{F} \subset \mathcal{U}(A)$

ENOUGH TO SHOW: $\exists\ v \in M$, ISOMETRY ST

$$\forall a \in \mathcal{F},\ \varphi(a) \sim_\varepsilon v a v^* \text{ IN } \|-\|_\chi^\#\text{-NORM.}$$

AS M SEMIDISCRETE, CAN ASSUME:

$$\begin{tikzcd} A \arrow[r, "\sigma"] \arrow[rr, bend right, "\psi"'] & M_n(\mathbb{C}) \arrow[r, "\eta"] & M \end{tikzcd} \qquad \sigma, \eta \text{ CP MAPS}, \quad \psi = \eta \circ \sigma$$

$$\forall a \in \mathcal{F},\quad \varphi(a) \sim \psi(a) \text{ IN } \|-\|_\chi^\#\text{-NORM.}$$

FIRST APPLICATION:

EXTREME AMENABILITY.

DEF: G TOP. GRP.

G EXTREMELY AMENABLE (EA) IF

ANY G-ACTION ON A COMPACT SPACE HAS

A FIXED PT.

REM: ① A LOC. COMPACT GRP IS NEVER EA.

[VEECH: A LOC. COMP. GRP HAS A FREE ACTION ON
A COMPACT SPACE]

② FIRST EXAMPLE BY HERER-CHRISTENSEN (1975)
FIRST "NATURAL" EXP. BY GROMOV-MILMAN ('83)

$\mathcal{U}(\ell^2(\mathbb{N}))_{SOT}$ EA.

[PF. USES NOTION OF CONCENTRATION OF
MEASURE ON HIGH-DIMEN. STRUCT.]

* [GLASNER; FURSTENBERG-WEISS].

$L^0([0,1], \mathbb{T})$, WITH TOPO. CONV. IN MEAS.

* [G., PESTOV]. (X, μ) NON-ATOMIC STD BOREL PROB SP.

  * $\mathrm{AUT}(X,\mu)$ F.M.P. TRANSF. WITH WEAK TOPOL.

    (TOP. INDUCED BY SOT ON $\mathrm{ISOM}(L^2(X,\mu))$).

  <u>REM</u>: $\mathrm{AUT}(X,\mu)$, WITH UNIF. TOP., NOT EA.

  (TOP. INDUCED BY $d(\alpha,\beta) = \mu\{x ; \alpha(x) \neq \beta(x)\}$)

  * M TYPE II OR TYPE III INJECTIVE FACTOR

    $U(M)$, WITH THE $s(M, M_*)$-TOPOL. IS EA.

  <u>REM</u>: (DE LA HARPE) M INJECTIVE IFF $U(M)$ AMEN.

  (PATERSON) A C^*-ALG.

  A NUCLEAR IFF $U(A)$, WITH $\sigma(A, A^*)$-TOP. AMENABLE.

  * A NUCLEAR IFF $U(A)$, WITH $\sigma(A, A^*)$-TOP. STRONGLY AMENABLE.

THM: M PROPERLY INF. INJECTIVE VN FACTOR

$\mathcal{U}(M)$ WITH THE $s(M,M_*)$-TOPOL., HAS THE

FIXED PT. PROPERTY ON COMPACT METRIC SP.

REM: ① FIXED PT. PROP. ON COMP. METRIC SP

$\Longrightarrow$ FIXED PT. PROP ON COMPACT SP.

HENCE, NEW PF. OF E.A. OF $\mathcal{U}(M)$.

SECOND APPLICATION

RECALL: (M,τ) TYPE II_1 FACTOR

ω FREE ULTRAFILTER ON $\mathbb{N}$.

$$M^\omega = \ell^\infty(\mathbb{N}, M) \Big/ \{ x=(x_n) \ ; \ \|x_n\|_2 \xrightarrow[n\to\omega]{} 0 \}$$

ULTRAPOWER M^ω IS A TYPE II_1 FACTOR.

[CONNES, 1976] Q?: EVERY FACTOR OF TYPE II_1

EMBEDS INTO AN ULTRAPOWER

R^ω.

LARGELY THROUGH KIRCHBERG'S WORK, MANY EQUIV. DEF. OF CONNES CONJECT. IN PART.,

<u>THM</u> (PESTOV–USPENSKIJ). TFAE

① CONNES' CONJECT. IS TRUE

② $C^*(F_\infty \times F_\infty)$ IS RESIDUALLY FIN. DIM.

③ $C^*(F_2 \times F_2)$ ———.

④ $\mathcal{U}(\ell^2)_{SOT}$ HAS THE KIRCHBERG PROP.

WHERE

<u>DEF.</u> A TOP. GRP G HAS THE <u>KIRCHBERG PROPERTY</u> IF $\{ h : F_\infty \times F_\infty \xrightarrow[\text{HOMO.}]{} G \ ; \ \overline{h(F_\infty \times F_\infty)} \text{ COMPACT} \}$ DENSE IN $\mathrm{REP}(F_\infty \times F_\infty, G)$.

<u>REM</u>: ① $\mathrm{REP}(F_\infty \times F_\infty, G) \subset G^{F_\infty \times F_\infty}$ PROD. TOP.

② (PESTOV–USPENSKIJ)

ISOM (URYSOHN METRIC SP), WITH COMPACT-OPE TOP. HAS THE KIRCHBERG PROP.

URYSOHN METRIC SPACE $\mathbb{U}$ CHARACTERIZED BY

1. $\mathbb{U}$ COMPLETE METRIC SP.
2. $\mathbb{U}$ CONTAINS AN ISOMETRIC COPY OF EVERY SEP. METRIC SP.
3. $\mathbb{U}$ IS ULTRAHOMOGENEOUS.

   (ie. EVERY ISOM. BETWEEN TWO FINITE SUBSP. EXTENDS TO AN ISOMETRY OF $\mathbb{U}$).

USING UNIQUENESS THM. AND IDEAS THEORY OF ABSORBING EXT.

THM TFAE TO CEC.

1. FOR EVERY INJECTIVE PROPERLY INFINITE FACTOR M, WITH M_* SEP., $\mathcal{U}(M)$ HAS THE KIRCHBERG PROP.
2. $\exists$ M AS ↑
3. FOR EVERY UNITAL SEP. SIMPLE NUCLEAR C^*-ALG. A, $\mathcal{U}(M(A \otimes K))$ HAS THE KIRCH. PROP.
4. $\exists$ A AS ↑

"LOCALIZED" VERSION OF HADWIN–VOICULESCU THM.

(2) SPECIALIZE THE CLASS OF C^*-ALG. :

* A UNITAL, SIMPLE, NUCLEAR, SEPARABLE
* REPLACE $B(H)$ BY A GEN. VNA M.

  PT-WISE NORM CONV BY – $\sigma(M,M_*)$-TOPO.

DEF: $\varphi, \psi : A \to M$ $*$-HOMO. ARE **w^*-APPROX. UNIT. EQ.** IF $\exists$ (u_α), (v_β) UNITARIES OF M ST. $\forall a \in A$,

$$u_\alpha \varphi(a) u_\alpha^* \xrightarrow[\sigma(M,M_*)]{} \psi(a) \quad \text{AND} \quad v_\beta \psi(a) v_\beta^* \xrightarrow[\sigma(M,M_*)]{} \varphi(a)$$

* REPLACE $B(H)$ BY C UNITAL, SEP., C^*-ALG.

  PT-WISE NORM CONV. BY. PT-WISE $\sigma(C, C^*)$-TOPO.

DEF: $\varphi, \psi : A \to C$ ARE **w-APPROX. UNIT. EQ.**

IF ....

2.1. M FINITE VNA.

THM: (DING-HADWIN). M FINITE VNA.

A SEP. NUCLEAR C^*-ALG.

$\varphi, \psi : A \to M$ INJECTIVE $*$-HOMOM. ST.

$\forall \tau$ NORMAL TRACIAL STATE ON M, $\forall a \in A_+$

$$\tau \circ \varphi(a) = \tau \circ \psi(a)$$

THEN φ AND ψ ARE w^*-APPROX. UNIT. EQ.

2.2. M PROPERLY INFINITE VNA.

THM: M PROPERLY INF. FACTOR, M_* SEP.

$1 \in A$ SEP., NUCLEAR C^*-ALG.

$\varphi, \psi : A \to M$ UNITAL, $*$-HOMOM. ST.

$$\mathrm{Ker} \circ \varphi = \mathrm{Ker} \circ \psi .$$

THEN φ AND ψ ARE w^*-APPROX. UNIT. EQUIV.

GENERAL CASE:

NEED TO USE REDUCTION FOR M VNA, M_* SEP.

NEED TO USE A SEP. FOR M GENERAL CASE.

THM: A UNITAL, NUCLEAR, SEP. C^*-ALG.

M VNA.

$\varphi, \psi : A \longrightarrow M$ UNITAL $*$-HOMOM. ST.

$\mathrm{RANK}(\varphi) = \mathrm{RANK}(\psi)$.

THEN φ AND ψ ARE w^*-APPROX. UNIT. EQ.

CONVERSELY:

DEF: A SIMPLE, UNITAL, SEP. C^*-ALG.

A HAS THE WEAK* UNIQUENESS PROPERTY

IF $\forall$ M VNA, $\forall$ $\varphi, \psi : A \to M$ UNITAL, INJECTIVE

SUCH $\mathrm{RR}(\varphi) = \mathrm{RR}(\psi)$, THEN φ AND ψ ARE

w^*- APPROX. UNIT. EQUIV.

THM: A UNITAL, SEPARABLE, SIMPLE C^*-ALG.

TFAE: ① A NUCLEAR

② A HAS THE WEAK*-UNIQUENESS PROP.

③ A HAS THE WEAK-UNIQUENESS PROP.

DEF: A HAS THE WEAK UNIQUENESS PROP. IF

$\forall$ C UNITAL, SEP. C^*-ALG.

$\forall$ $\varphi, \psi : A \to C$ UNITAL, INJECTIVE $*$-HOMO.

ST. $\mathrm{Re}(\varphi) = \mathrm{Re}(\psi)$

THEN φ AND ψ ARE ω-APPROX. UNIT. EQ.