Not the Nottingham talk

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(The slides, in particular the Cuntz semigroup part, have been slightly modified. Thanks to Leonel Robert, Aaron Tikuisis and Andrew Toms for convincing me to use the compact containment relation.)

Elliott program as of 2003

All nuclear, separable, simple, unital C*-algebras are classified by the Elliott invariant,

$$((K_0(A), K_0(A)^+, 1), K_1(A), T(A), \rho_A).$$

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Ell}(A) \\ \downarrow & & \downarrow \\ B & \longrightarrow & \mathrm{Ell}(B) \end{array}$$

Rørdam, Toms, 2004 - counterexamples.

New directions

1. Classification of nuclear, simple, unital, separable, $\mathcal{Z}$-stable C*-algebras.
2. Cuntz semigroup as an invariant.

Descriptive set theory: Abstract classification

Assume the collection X of objects we are trying to classify forms a ‘nice’ space, typically a Polish space or a standard Borel space and the equivalence relation E is a Borel or analytic subset of X^2. (*Analytic* set is a continuous image of a Borel set.)

The basic concept of abstract classification

Definition

If (X, E) and (Y, F) are equivalence relations, E is *Borel-reducible* to F, in symbols

$$E \leq_B F,$$

if there is a Borel-measurable map $f : X \to Y$ such that

$$x \, E \, y \Leftrightarrow f(x) \, E \, f(y).$$

The intuitive meaning:
(1) Classification problem represented by E is at most as complicated as that of F.
(2) F-classes are complete invariants for E-classes.

Example

Spectral theorems.

Glimm–Effros Dichotomy

If $E \leq_B =^{\mathbb{R}}$ we say E is *smooth*.
For x, y in $2^{\mathbb{N}}$ let

$$x \, E_0 y \text{ iff } (\exists m)(\forall n \geq m) x(n) = y(n)$$

Theorem (Harrington–Kechris–Louveau, 1990)

If E is a Borel equivalence relation on a Polish space then either E is smooth or $E_0 \leq_B E$.

Theorem (Glimm, 1960)

If A is separable then either the equivalence of pure states on A is smooth or E_0 is Borel-reducible to it.

The proof of HKL theorem made heavy use of metamathematics.
An 'elementary' proof was found only in 2009 by Ben Miller.

Modelling classification problems I

Example (The Polish space of countable groups)

A countable group G is coded by
$(\mathbb{N}, e_G, x_G, {}^{-1}{}_G)$, for $e \in \mathbb{N}$, $\times_G : \mathbb{N}^2 \to \mathbb{N}$, ${}^{-1}{}_G : \mathbb{N} \to \mathbb{N}$.

This is a closed subspace of the compact metric space
$\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}^3) \times \mathcal{P}(\mathbb{N}^2)$.

The isomorphism $\cong^G$ is an S_∞-orbit equivalence relation.

Modelling classification problems II

In general, a given concrete classification problem for category $\mathcal{C}$ is modelled by a standard Borel space (X, Σ) and $F\colon X \twoheadrightarrow \mathcal{C}$ such that the relation E on X,

$$x \, E \, y \Leftrightarrow F(x) \cong F(y)$$

is analytic (i.e., a continuous image of a Borel set).

Classification by countable structures

An equivalence relation (X, E) is classified by countable structures if there is a countable language L and a Borel map from X into countable L-models such that

$$x \, E \, y \text{ iff } F(x) \cong F(y).$$

This is equivalent to being $\leq_B$ an S_∞-orbit equivalence relation.

Lemma (Sasyk–Törnquist 2009, after Hjorth)

If $G \subsetneq F$ are separable Banach spaces, G is dense in F, and $\mathrm{id}\colon G \to F$ is bounded, then the coset equivalence F/G cannot be classified by countable structures.

Example

c_0/ℓ_2.

Examples

Theorem (Kechris–Sofronidis, 2001)

Unitary operators up to conjugacy are not classifiable by countable structures.

Theorem (Foreman–Weiss, 2004)

Ergodic measure-preserving transformation up to conjugacy are not classifiable by countable structures.

Theorem (Sasyk–Törnquist, 2009)

Type II_1 factors are not classifiable by countable structures. The same result applies to II_∞ factors and III_λ factors for $0 \leq \lambda \leq 1$, to injective III_0 factors and to ITPFI factors.

Effros Borel space

For a Polish space X let X^* be the space of closed subsets of X. The σ-algebra Σ on X^* is generated by sets

$$\{A \in X^* : A \subseteq U\}$$

where U ranges over open subsets of X.

Proposition

(X^*, Σ) *is a standard Borel space. If* X *is a separable C*-algebra then*

$$S(X) = \{B \in X^* : B \textit{ is a subalgebra of } X\}$$

is a Borel subspace of X^**.*

Examples

Theorem (Kirchberg, 1994)

$S(\mathcal{O}_2)$ *is the space of all exact separable C*-algebras.*

Theorem (Pisier–Junge, 1995)

$S(A)$ *is not the space of all separable C*-algebras for any separable C*-algebra* A*.*

Borel space of separable C*-algebras

Definition (Kechris, 1996)

Let Γ *be* $\mathcal{B}(\ell_2)^{\mathbb{N}}$*, with respect to the weak operator topology. Then*

$$\Gamma \ni \gamma \mapsto C^*(\gamma)$$

maps Γ *onto the space of all separable C*-algebras represented on* H*, and*

$$\gamma_0 \, E \, \gamma_1 \Leftrightarrow C^*(\gamma_0) \cong C^*(\gamma_1)$$

is analytic.

There is also a space Δ of abstract separable C*-algebras.

Two representations are *equivalent*.

Lemma (Kechris, 1996)

There are Borel maps $\Phi_j \colon \Gamma \to \Gamma$ *(*$j = 1, 2, 3$*) such that*

1. $\Phi_1(\gamma)$ *enumerates a norm-dense subset of* $C^*(\gamma)$*,*
2. $\Phi_2(\gamma)$ *enumerates a norm-dense subset of* $C^*(\gamma)_+$*,*
3. $\Phi_3(\gamma)$ *enumerates a norm-dense subset of the projections of* $C^*(\gamma)$*.*

Proposition (Effros, 1996)

The set $\{\gamma \in \Gamma : C^*(\gamma) \text{ is nuclear}\}$ *is Borel.*

Classification problem of C*-algebras

Lemma (Farah–Toms–Törnquist)

There is a Borel map $\Psi\colon \Gamma^2 \to \Gamma$ *such that*

$$C^*(\Psi(\gamma_1,\gamma_2)) \cong C^*(\gamma_1) \otimes_{\mathrm{min}} C^*(\gamma_2).$$

Proposition (Farah–Toms–Törnquist)

Computation of the Elliott invariant is Borel.

Theorem (Farah–Toms–Törnquist)

The isomorphism of separable, simple, unital, nuclear C-algebras is not classifiable by countable structures.*

Actually we can do this for AI algebras.

Classifiable C*-algebras are not classifiable... by countable structures

Theorem (Elliott, 1993)

AI algebras are classified by the Elliott invariant.

Theorem (Farah–Toms–Törnquist)

If L is a countable language, then the isomorphism of countable L-models is $\leq_B$ *to the isomorphism of AT algebras.*

The top

Theorem (Ferenczi–Louveau–Rosendal, 2009)

Isomorphism of separable Banach spaces is the $\leq_B$-maximal analytic equivalence relation.

In particular, separable Banach spaces cannot be classified by orbits of a polish group action.

Theorem (Kechris–Solecki, 200?)

Homeomorphism of compact metric spaces is $\leq_B$ a Polish group action.

Below group action

Proposition (Farah–Toms–Törnquist)

The isomorphism of simple separable nuclear C-algebras is $\leq_B$ to an orbit equivalence relation of a Polish group action.*

Pf. (The unital case.) Consider the Effros Borel space of subalgebras of $C \cong \mathcal{O}_2$ and the natural action of $\mathrm{Aut}(C)$ on it. For nuclear, separable, simple, unital A we have $C \cong A \otimes \mathcal{O}_2$ (Kirchberg).

$$\text{Borel map:} \qquad A \mapsto F(A)$$

where $F(A)$ is a subalgebra of C isomorphic to A such that $C = F(A) \otimes \mathcal{O}_2$.
Then $A \cong B$ if and only if there is an automorphism of C sending $F(A)$ to $F(B)$. $\square$

Borel space of Cuntz semigroups

(The following space has the space of Cuntz semigroups of separable C*-algebras as its proper subspace, but there is no precise range of invariant result anyway.)
Let **Cu** be the space of all countable ordered semigroups with 0 and distinguished subset of elements compactly contained in themselves. This is a compact subspace of $\mathcal{P}(\mathbb{N})^6$.

Proposition

There is a Borel map $\Phi\colon \Gamma \to \mathbf{Cu}$ *such that the equivalence relation* E *on* $\mathbf{Cu}$

$$\Phi(\gamma)\,\mathrm{E}\,\Phi(\gamma') \text{ if and only if } \mathrm{Cu}(C^*(\gamma)) \cong \mathrm{Cu}(C^*(\gamma'))$$

is analytic.

Sketch of the proof: $\gamma \mapsto \mathrm{Cu}(C^*(\gamma)$ is Borel

Replace $\gamma \in \Gamma$ with γ' such that

$$C^*(\gamma') \cong C^*(\gamma) \otimes \mathcal{K}.$$

Let (p_n) be a fixed enumeration of all *-polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. Maps

$$\Gamma \ni \gamma \mapsto (p_n(\gamma)) = x_n \in \Gamma$$
$$\Gamma \ni \gamma \mapsto (p_n(\gamma) + p_n(\gamma))^*)_+ = a_n \in \Gamma$$

are Borel, and they give effective enumerations of dense subsets of $C^*(\gamma)$ and of $C^*(\gamma)_+$, respectively. Modify (a_n) so that a_n is a projection iff n is even.

Sketch of the proof: $\gamma \mapsto \mathrm{Cu}(C^*(\gamma)$ is Borel, II

Define $\precsim_\gamma$ on $\mathbb{N}$ by

$$m \precsim_\gamma n \qquad \Leftrightarrow \qquad (\forall i)(\exists j)\|x_j a_n x_j^* - a_m\| < 1/i$$

The map $\Gamma \ni \gamma \mapsto \precsim_\gamma \in \mathcal{P}(\mathbb{N})^2$ is Borel.

Map $\Gamma \ni \gamma \mapsto \oplus_\gamma \in \mathcal{P}(\mathbb{N})^3$ is similarly Borel...
...and so is $\Gamma \ni \gamma \mapsto \ll_\gamma \in \mathcal{P}(\mathbb{N})^2$ (here $\ll$ is the compact containment relation).

The quotient structure on $\mathbb{N}$, wrt $\approx_\gamma$, $\oplus_\gamma$ and $\precsim_\gamma$, belongs to **Cu**.

Sketch of the proof: $\gamma \mapsto \mathrm{Cu}(C^*(\gamma))$ is Borel, III: Recovering Cu

Given $D = (\mathbb{N}, \precsim, +, C) \in \mathbf{Cu}$, define $\tilde{D}$ and $D^{\nearrow}$ as follows.
Let $D^{\nearrow}$ be the set of $\ll$-increasing sequences in D, and let

$$(x_n) \leq (y_n) \quad \Leftrightarrow \quad (\forall m)(\exists n) x_m \ll y_n \text{ and } y_m \ll x_n$$

Let $(x_n) \sim (y_n)$ iff $(x_n) \leq (y_n)$ and $(y_n) \leq (x_n)$.
Then $\tilde{D} = D^{\nearrow} / \sim$ is the Cuntz semigroup of $C^*(\gamma)$.

Sketch of the proof: $\gamma \mapsto \mathrm{Cu}(C^*(\gamma))$ is Borel, III
Isomorphism relation is analytic

Lemma

The relation on **Cu** *defined by* $D_1 \mathrel{\mathsf{E}} D_2$ *iff* $\tilde{D}_1 \cong \tilde{D}_2$ *is analytic.*

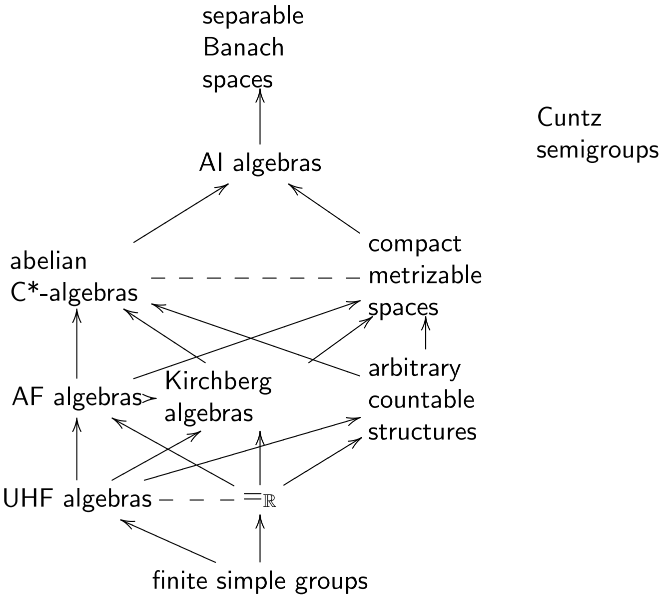
Proof.

$\tilde{D}_1 \cong \tilde{D}_2$ iff:

$$\begin{aligned} (\exists \Phi_1)\colon D_1 \to D_2^{\nearrow}, &\ \text{homomorphism} \\ (\exists \Phi_2)\colon D_2 \to D_1^{\nearrow}, &\ \text{homomorphism} \\ \Phi_1 \circ \Phi_2 = \mathrm{id}_{D_2} / \sim \\ \Phi_2 \circ \Phi_1 = \mathrm{id}_{D_1} / \sim \end{aligned}$$

□

Relative complexity of some isomorphism relations



Theorem (Farah–Toms–Törnquist)

Biembeddability relation E_{bi} of separable AF algebras is not classifiable by countable strucutures. Moreover, any K_σ equivalence relation is $\leq_B E_{bi}$.

Some problems

Question

Is the isomorphism of all separable, unital C-algebras $\leq_B$ orbit equivalence relation?*
What about the not necessarily simple nuclear C-algebras? Exact C*-algebras? Arbitrary C*-algebras?*

Problem

Develop set-theoretic framework for Elliott's functorial classification.

Problem

Is the isomorphism of countably determined Cuntz semigroups $\leq_B$ orbit equivalence relation?