Practical Optimization in Finance, Fall 2001

“Aftermath” of Lectures on Wednesday October 31, 2001

Projects: Project 2 has now been posted on the homepage. It deals with pricing in binomial models, and is not intended to be particularly hard. The “hand-in”-deadline is November 15. Feel free to use any software you like.

The Software Many of my numerical illustrations will be done in the spreadsheet Excel. This is not an attempt to “sell” Excel1 – as most academics I have little enthusiasm for Microsoft – but rather to minimize effort for myself. For that reason, it could easily happen that I suddenly change to “something else”. Anyway, if you want an introduction to Excel (“the basic stuff”), you want it fast, and you want it cheap, my advice advice is to search the www. I did a search in Google for “Introduction to Excel” and almost 10,000 hits.

The Lectures October 31: Project 1-answers were handed back (and forth). I repeated the basics of pricing by no arbitrage in binomial models. I won’t do that here; look at the material on the course homepage or come to the lectures last week!

Then we looked at how to specify models: what $u$ and $d$ values should we use? A suggested parametrization for some given $\Delta$ was $p = \frac{1}{2}$ and

$$ u_\Delta = \exp(\mu \Delta + \sigma \sqrt{\Delta}), \quad d_\Delta = \exp(\mu \Delta - \sigma \sqrt{\Delta}), $$

for some yet-to-be-determined constants $\mu$ and $\sigma > 0$. We noted that the logarithmic rates of return (simply log-returns in the following),

$$ r_\Delta(t) = \ln(S_t/S_{t-\Delta}), $$

satisfy that $\mathbb{E}^\mathcal{P}(r_\Delta) = \mu \Delta$ and $\text{Var}^\mathcal{P}(r_\Delta) = \sigma^2 \Delta$, and are independent. (This means that we have $\mathbb{E}^\mathcal{P}(r_{n\Delta}) = \mu n \Delta$ and $\text{Var}^\mathcal{P}(r_{n\Delta}) = \sigma^2 n \Delta$ for any $n \in \mathbb{N}$.)

If we then collect historical data on stock-prices $S^i_t$, say $N + 1$ observations that are each $M$ periods of length $\Delta$ apart, calculate observed log-returns and put $\hat{r} = \frac{1}{N} \sum_{n=1}^N r^i_{n\Delta}$, then good estimators of $\mu$ and $\sigma$ are (use basic statistics)

$$ \hat{\mu} = \frac{\hat{r}}{M \Delta} \quad \text{and} \quad \hat{\sigma} = \frac{\sqrt{\frac{1}{N} \sum_{n=1}^N (r^i_{n\Delta} - \hat{r})^2}}{M \Delta}, $$

I showed a spreadsheet with a numerical example. You can get it on the homepage. If $M$ is large (and the “true” $\Delta$ is small), then it can be shown by a Central Limit Theorem argument that the observed log-returns should be approximately normally distributed. This looked mildly plausible for monthly returns. There are several choices/adjustment one has to make in practice. For instance regarding $M$ and $N$, whether to give more weight to recent observations and how many days there are in a year! (Empirical evidence suggests 250; or in other words that only days when the stock-exchange is open matter.)

But we are interested in option pricing, hence $\mu$-probabilities. These can also be calculated. You will work with that in Project 2, where you’re in for a surprise (especially for small $\Delta$) when it comes to $\mu$’s influence on option prices. I don’t want to give things away, but it does show why there’s no loss of generality, really, in the Cox-Ross-Rubinstein-parametrization of $u$ and $d$ (mentioned/used in R&P).

Next on the agenda was portfolio choice for an investor. A bit different, sure, but since we can now build empirically reasonable binomial models, why not look at it? Suppose we have an $N$ period binomial model (with time-step length $\Delta$), and an investor who focuses on his wealth at time $T = N \Delta$. He has a concave criterion function he wants to optimize; in economics we’d say “he maximizes expected utility”. He has a fixed amount of money, $W(0)$, to invest in stock or bank-account, but it is only possible for him to trade at time $t$. What should he do? “In math”, we assumed the faces the problem

$$ \max_{x_0} \mathbb{E}^\mathcal{P}(u(W(T))) \quad \text{st} \quad x_0 S(0) + x_0 \leq W(0) $$

$$ x_0 S(T) + x_0 (1 + r)^N = W(T) $$

$$ x_0, x_0 \geq 0, \quad \text{(ie. no short-selling)} $$

where $u$ has the form

$$ u(x) = \frac{x^\gamma - 1}{\gamma}, $$

for some $\gamma \leq 1$. If $\gamma = 1$, the investor is risk-neutral, $\gamma < 1$ means risk-aversion (and a non-linear criterion function). (The case $\gamma = 0$ means $u(x) = \ln(x)$ by a limiting argument. There’s no problem in $\gamma > 0$, but for fixed $x > 0$, $u(x) \approx -1/\gamma \approx 0$ when $\gamma$ is “very negative”, so …) Clearly, all of the
initial wealth is invested, so the problem is essentially one-dimensional. So we could try to analyze it theoretically with Lagrange-techniques and the like. But rather, we did things numerically with Excel to get a feel for the problem. If we want to look at larger, multi-period portfolio choice problem, we'd better understand the simple ones. Further, we saw Excel's "Solver" routine in action. (Click Tools → Solver, and the rest is self-explanatory. For space-saving reasons, the Solver package may not immediately appear on the menu. Then you need to click Tools → Add-Ins and install it. And while you're at it, installing the Analysis ToolPak package is a good idea.) You can find the Excel-files on the homepage; do try to play around with them. A few things are worth remembering:

- Actual probabilities ($p$'s) matter when we can't trade continuously.
- A risk-neutral investor puts all his money in whatever gives the highest expected rate of return. (And note: Its expected (discrete) rates of returns over the $N$ periods - not log-returns - that naturally define the "break-even" point.)
- A risk-averse investor may choose to diversify his portfolio, i.e. to invest in both stock and the bank-account. (But note that if the expected rate of return on the stock is high enough, a risk-averse investor will also only buy stock. The catch-phrase here is "There a difference between being risk-averse and being stupid!!".)

Wednesday November 7: I'll put some more "bells & whistles" on the portfolio optimization problem (options, "certainty equivalents", dynamic trading (possibly)). Then I'll say something about the limiting case $\Delta \to 0$. This gives the famous Black-Schoes model (and formula). No-one should take a finance-related course and not see this. And then I'll illustrate how the binomial no-arbitrage approach can be used to build models for/with stochastic interest rates, aka term structure models.

Kindly,

Rolf