Planning Your Own Debt

Søren Nielsen and Rolf Poulsen

September 20, 2001

Abstract

We model the Danish mortgage market for mortgage-backed securities with a two-factor interest rate model and use a stochastic programming approach to analyze how an individual homeowner should initially compose and subsequently refinance his mortgage in an optimal way. Results show that the "rules of thumb" used by financial institutions are reasonable, although not suited for more aggressive homeowners, for whom the delivery option is of some value. More risk-averse investors should also benefit frequently, but use more diversified portfolios. Results are sensitive to whether a one- or two-factor model is used, provided the former is suitably calibrated.

Keywords: Term structure of interest rates, mortgage-backed securities, portfolio choice, stochastic programming.

JEL-classification: C61, D81, E43, G11, G21.

1 Introduction

Spurred on by the product innovation and advice of financial institutions, many Danish home-owners/mortgagees manage their debt very actively. The individual mortgagee faces a number of non-trivial decisions. He has to decide whether to use adjustable rate loans where the debt is refinanced on a yearly basis (or every 3 or 5 years) or the more traditional fixed rate 30- or 35-year mortgages (or some combination of). The mortgagee has also options. There is a call-feature because the borrower can pay back the remaining principal at any time, and there is a delivery option because he can buy back the debt at market value (a feature not present in mortgages in the U.S.). The objective is to model the problem faced by a Danish mortgagee who wants to initiate and maintain a mortgage in an optimal way. We use a stochastic two-factor term structure model (calibrated to market prices and volatilities of non-callable bonds and to prices of callable bonds) and formulate the problem of optimal (with respect to some concave criterion) debt management (subject to investor specific tax rates and transaction costs) as a multistage stochastic programming problem that can be solved with standard software.

The modelling of mortgage prepayment behavior has created much debate literature, for instance Richard A. Roll (1989), Kang & Zenios (1992), and Jakobsen (2001) for a Danish perspective. Portfolio management with mortgage backed securities (MBS) as seen from the investor's side has also been addressed, e.g., by Zenios (1995). When the problem is viewed from the mortgagee's side, an option-pricing approach (with some modifications) is usually taken, as in the classic Stannum (1995) for example. A large concern in that literature is whether aggregating the micro-economic analysis helps explain the overall prepayment behavior in the market (a short survey would be "it is doubtful"). The focus will be mainly on how an individual mortgagee should compose and manage his portfolio. This study specifically addresses the portfolio choice problem of the mortgagee. Prevalent "tests" have mostly been of "(20/20) hindsight" type. Given the historical prices, what would it have optimal to do, i.e. the analysis is based on a simulation with no attention to risk.

The outline of the rest of the paper is as follows. In Section 2 we give a more detailed description of the Danish mortgage market, with particular emphasis on the products available to the individual mortgagee and the choices/trade-offs he faces. Section 3 introduces a stochastic interest rate model. We use a Canadian two-factor model for zero coupon bond yields formulated in the Heath-Jarrow-Morton framework. In this section we also propose a simple, but very operational, regression approach for determining prices of callable mortgage backed bonds from zero coupon bond yields. In Section 4 the problem that the mortgage from Section 2 faces and the dynamic model for interest rate and MBS prices from Section 3 are combined in a multistage stochastic programming formulation. Section 5 gives numerical results. The results show that the "rules of thumb" used by financial institutions are
reasonable, although least suited for more aggressive mortgages, for whom the delivery option is of prime value. More risk-averse investors should also avoid portfolios longer than the mortgage period. It appears that results are insensitive to whether a one- or two-factor model is used, provided the factors are suitably calibrated. Section 6 concludes and outlines topics for future research.

2 The Danish Mortgage Market and the Individual Mortgagor

The interest rate policy of the Danish Central Bank usually mimics that of the European Central Bank's short rates, and the yields on government bonds also closely follow those in "Euroland," with the addition of a spread of 25 to 50 basis points. On several occasions the Danes have voted "No" to joining the single European currency, the Euro, but still the exchange rate is quite stable (between 7.40 and 7.50 DKK will buy you 1 Euro).

Though small in absolute terms, the Danish mortgage market has some interesting features. Notably, mortgages have been financed by 2Y- or 3Y-fixed-rate annuities that were issued through intermediaries (first only dedicated mortgage companies since 1970 also banks) in a liquid market. With loans of this type the mortgagor can change the notional principal at any time, to other words the bonds are callable. When interest rates have dropped the mortgagor can issue new debt at these lower rates, typically in the form a new callable bond with lower coupon rate, and use the proceeds from this to pay off the old mortgage.

Due to various frictions these bond's prices do actually increase above par without being called. One major primary reason is the inefficiency of mortgage's behavior. It is not possible or worthwhile for the individual mortgagor to be highly efficient in this respect. But there is also a typical option aspect: Even when an option is in-the-money it may be too small to be valuable, but may become even better yet.

Conversely, if the interest rates increase the mortgagor can buy back the, now relatively cheap, mortgages in the market, again funding this by issuing a new mortgage, now with a higher coupon rate. This will make the installments larger, but reduce the principal on the mortgage, and the mortgagee has purchased himself better inflation rates. Mortgage rates generally drop (and bound the market value of the debt). This was a strategy widely recommended by financial institutions from mid-1999 to mid-2000, because there was a perception that "rates are high" (since they had recently gone up) and "they will come down" (in particular because the common belief was that Denmark would soon join the Euro). Further, this strategy has a "lose-lose" since a large proportion of the installments are inelastic payments that are deductible. (And tax asymmetries between typical borrowers and lenders mean that "tax gains" are not fully offset by lower prices.) A good refinancing strategy can be to issue a higher mortgage, but allow lower rates and lower, in other ways loaning bonds in closer to par as possible. (As pointed out, the prices of callable bonds do go up above par, but mortgages are by law only allowed to issue when the price is at or below par.)

A key question is by how much rates should change before refinancing is optimal. Danish banks now offer so-called mortgage-watch programs, where they alert their customers to opportunities. Of course the banks make a sizeable profit. There is a fixed cost to the bank to run typically DKK 2,000-3,000 and a variable cost of 0.15% of the price of the new issue when a customer re-finance (in addition to which some taxes are to be paid), so it is in their interest to induce refinancing.

To summarize, the purchase of MBSs, the credit risk is very low because the lending institution pools and securizes the loans. But there are other issues that make the purchase attractive, in particular a high yield for the investors in the bond structure. Note the asymmetry: the mortgagor takes in accounts very specifically his own characteristics, whereas the purchaser buys the "average" mortgage. Ideally, one should consider both sides of the market and then arrive at (probably) a market price by some equilibrium argument. However, we model only adds to the side of the market, the overall market behavior is exogenous, but modeled a stochastic system in a mathematically plausible way.

In the mid 90's, adjustable rate mortgages (ARMs) in the form of revolving short-term loans in a variety of forms were introduced following a legislative change. These loans were first offered by RealCom Danmark® then (reluctantly) by all other intermediaries. The simplest one is the so-called FL, whereby the mortgage principal is refinanced

As long as the price is below par you can in fact take in an offer — valid for 36 months — from a bank or mortgage institution that allows you to remortgage bonds at the prevailing market price.

RealCom Danmark called their product First off, a term that quickly became synonymous ARM. And much to the dismay of the rest of the financial sector, RealCom Danmark succeeded in getting it approved as a registered instrument.
every year in January at the prevailing short rate. Another typical one is the F33, whereby a third of the outstanding principal is refinanced every year at the 3-year rate many of its possibilities exist. They all share two characteristics: They depend on the short end of the yield curve, and they are vulnerable to increases in short rates. With a normal, positively sloped term structure they have some appeal compared to the long-term bonds, & the market for these bonds, after some hesitation, was very large. The bonds are now also offered with various caps, making them very similar to U.S. style ARMs. Figure 1 shows the Danish short and long rates during the period 1997 to early 2001. Although ARMs had an obvious appeal during the early years, there was a significant narrowing in 2000 due to the uncertainty about the Danish referendum Sep. 28 whether to join the Euro (see didn’t), making the situation far less obvious.

3 A Two-Factor Interest Model and Prices of Mortgage Backed Securities

Our model of the market has two components: A classic term structure model for the stochastic movement of zero-coupon bond 2CB P(T) prices and a model that links the ZCB yield curve (simply referred to as the term structure) to marketable MBS mortgages, which are (among) the relevant instruments for financing mortgages.

3.1 A Two-Factor Term Structure Model

Pitting today’s term structure is most easily done when we work in the Heath-Jarrow-Morton framework [see Heath, Jarrow & Morton (1992)]. Models in this framework are usually formulated in terms instantaneous forward rates,

\[ f(t, T) = \frac{\partial \ln P(t, T)}{\partial T}, \]

i.e., \( f(t, T) \) is the forward rate seen at time \( t \) for an extremely short term loan at time \( T \). A specification of the dynamics of all ZCB prices or yields is then octically equivalent, see Björk (1998, Chapter 15) for a lucid exposition of different approaches to continuous-time interest rate modelling.

We use the “Gaussian volatility” specification

\[ df(t, T) = \alpha(t, T)dt + \sigma_1 dW_1 + \sigma_2 e^{-\lambda(T-t)/2} dW_2, \]

where \( W_1, W_2 \) are (uncorrelated) Brownian motions under some probability measure \( P \) (the actual or subjective statistical physical measure), and \( \sigma_1, \sigma_2 \) and \( \lambda \) are positive constants.

This two-factor model allows for a wide range of term structure deformations than the (effectively) parallel shifts of a one-factor model. Figure 1 depicts the Danish 1Y, 3Y, and 30Y ZCB yields* for the period September 1997 to January 2001. Clearly, there would be poorly fitted by a model that allowed only parallel shifts. The form of the volatility is specified such that it is easy to incorporate the empirically robust fact that long rates are less volatile than short rates; a fact caused by mean-reversion. (The model may not allow a so-called volatility bump, where 1Y-3Y rates are the most volatile, but we see no evidence of that in our data as indicated in Figure 2.)

We want the model to be arbitrage-free, and from Heath et al. (1992) we know that this

---

*ZCB yield curves were estimated from prices of government bonds by the non-parametric smoothing technique described in Tanggaard (1997). The data were kindly supplied by Danmarks.
mean that there exists a 3-dimensional stochastic process $\tilde{\phi}$ such that the drifts of the forward rates in Equation (1) obey the so-called HJM-diffusion condition:

$$a(t, T) = \sigma_{f}(t, T) \int_{t}^{T} \sigma_{f}(t, s) ds - \sigma_{f}^{2}(t, T) \mu(t).$$

(2)

Assuming constant risk-premia $\tilde{\phi}(t) = (\phi, \phi)$, we can combine equations (1) and (2), and after some lengthy calculations arrive at an explicit expression for the term structure of the term

$$V = \left\{ v(t, T) = \int_{t}^{T} f(t, s) ds - A(t, s) \right\},$$

(3)

where $A$, $B$, and $Z$ are deterministic functions and $Z = (Z_{t}, Z_{s}) \sim N(0, I)$ is independent of the interest rate. The ZCB yield with time $t$ to maturity is defined by $y(t, r) = \ln P(t, t + r)$, so some further calculations on (3) we find the ZCB yield dynamics to be

$$dy(t, r) = dt + \sigma_{d} dW_{t} + \frac{2}{\tau} (1 - e^{-\sigma^{2}/2}) dW_{s}.$$

(4)

We use this to estimate the volatility related parameters $\sigma_{d}$, $\sigma_{f}$, and $\lambda$. This can be done reliably from high-frequency sampled historical data (without the need for an immense long time period from first to last observation). Suppose we have time series observations (at $t_{i}$'s that are $\Delta t = 1$ day or 1 week apart) on ZCB yields for maturities $\tau$ (3M, 6M, 1Y, 2Y, ..., $2Y$)

$$\text{std. dev.} \left( \frac{\Delta y(t_{i}, \tau)}{\sqrt{\Delta t}} \right) = \sqrt{\sigma_{d}^{2} + \frac{\sigma_{f}^{2}}{\tau} (1 - e^{-\sigma^{2}/2})}.$$

When $\Delta t$ is small the term in $\sigma_{d}$ is small compared to the $dW_{s}$-term, and besides the drift is not really dependent (as before the volatility is not). To determine volatility parameter $\sigma_{d}$ and $\sigma_{f}$ we estimate at the values that produce the best fit on average (for September 1997 and converted into $dW_{s}$ and $dW_{s}$ with volatility parameter $\lambda$ fixed) to the shape of the term structure at the 1Y and 25Y year points.

<table>
<thead>
<tr>
<th>$\sigma_{d}$</th>
<th>$\sigma_{f}$</th>
<th>$\lambda$</th>
<th>$\phi_{1}$</th>
<th>$\phi_{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0067530</td>
<td>0.021555</td>
<td>7.90573</td>
<td>0.063574</td>
<td>-0.4970</td>
</tr>
</tbody>
</table>

Table 1: Estimates of ZCB curve parameters

Figure 2: Theoretical (full line) and observed (*) standard deviations of Danish ZCB yields, when the Gaussian two-factor model is fitted by least squares to daily ZCB yield changes from September 1997 to January 2001.

The observed standard deviation and the fitted curve are shown in Figure 2. We see a fairly good fit; in factor analysis terms the model explains about 90% of the variation in the data.

As input to the optimization problem described in the next section, we need a discretized version of the model given by (3). To get this we use a binomial tree for the $(Z_{t}, Z_{s})$, and when we use the $\alpha$, “middle” and “down” values

$$\left( 1/\sqrt{2}, \sqrt{3/2}, \right), \left( -2/\sqrt{2}, 0 \right), \text{and} \left( 1/\sqrt{2}, -\sqrt{3/2} \right),$$

then the first two moments fit uncorrelated $N(0, 1)$-variables. Because we have chosen a HJM-model that is in fact Markovian (the log calculations show it), it also follows directly from, say, Ritchken & Sankaranarayanan (1995), it is possible to represent the discretized model in a lattice. But since the optimization problem we want to solve has path-dependent features, our actual implementation is not reconstructing. The resulting tree (see Figure 4) corresponds to a $N$-period stochastic optimization program with decision
points at \((t_0, t_1, ..., t_{n-1})\) and a final (trivial) horizon decision at \(t_N\). The tree has a total of \(2^N\) nodes in its non-recombining variant.

### 3.2 Market Prices of Callable Bonds

Of course market participants know full well that MBS annuities are callable. Hence they do not trade at prices that are the products of the non-callable annuity discounted by ZCB rates, but at lower prices to reflect the value of the embedded option. One approach would be to use "standard" option pricing techniques to determine the value of the embedded call-option (for different underlying bonds). This is numerically difficult (we have a two-factor model) and tends to have little explanatory power for actual mortgage behaviour and prices of MBSs. Another option is to set up a statistical model that links interest rates to observed prepayment behaviour and then payments to actual prices. Models of this type do a decent job of explaining observed behaviour, but are less satisfactory when it comes to explaining the pricing of MBSs, as discussed for the Danish market by Jakobsen & Sørensen (2000). Further, they are quite data-intensive (collecting & ordering prepayment information is hard work). So we suggest using the following simple regression approach. At dates \(t_i (01/30/1997, 01/30/1998, ..., 01/30/2000)\) and for coupon rates \(C_i (5.6, 7.0)\) we observe market prices of in-mo MBSs \(P_{MBS}(t_i; C_i)\) as well as the term structure, i.e. the mapping of time to maturity into ZCB rates, \(r \rightarrow y(t_i; + r)\). (In practice we have only a finite number of \(r_i\) but interpolate if we have to.) We find the vector of cashflows of the corresponding non-callable annuity, \(z(T_j, C_j)\) at date \(T_j\), and calculate a spread\(^9\).

We regress the logarithm of the spread on a constant, the coupon rate, the log rate \(y(t_i; 30Y)\) and the skew rate \(y(t_i; 1Y)\) i.e.

\[
\log(s(t_i; C_i)) = \alpha + a_1 C_i + a_2 y(t_i; 1Y) + a_3 y(t_i; 30Y) + \text{no}\text{.err}
\]

Evidently, we expect spreads to depend on the coupon rate (just as option prices depend on strike) and interest rate ("the underlying"), we use short and long rates for consistency with the two-factor model, while the constant can be interpreted as liquidity (and credit) spread.

Using log-regression the regression makes sure the spread is always positive and gives a better fit. We use the estimated spread/skew relation (for the specific time to maturity) to

\(^9\) Note that this is not what is commonly meant by an option adjusted spread (OAS).

The calculation of OAS requires a model where expected cashflows of the callable MBS can be found. Formally, \(\text{OAS} = \mathbb{E}\left[ \sum_{T_i} \text{cashflows}(T_i) \times e^{-\int_{T_i}^{T_N} y(t; 30Y) dt} \right] \), see Chen (1996) for more on OAS.

![Figure 3: Observed and predicted spreads for 5%, 6%, and 7% 30Y annuities.](image)
4 The Multistage Stochastic Programming Model

We now present a multistage, stochastic optimization model which implements the mortgagee's problems of initially establishing an optimal portfolio (of bonds), and also manages this portfolio optimally. Stochastic programming is a technique commonly used in operations research to solve large, complex and possibly non-linear optimization problems. Bére & Louveaux (1997) for instance. As many other modern OR techniques, it is surprisingly rarely used in finance. As is typical of investment problems, the "portfolio" in question will consist of just a single instrument in the risk-neutral case, but the model allows for risk-aversion by a suitable choice of objective. At this point we take for given the structure of the scenario tree (i.e., a trinomial tree such as shown in Figure 4), where at each node a "universe" of securities are available. In our case these securities are the bonds that the homeowner has available for refinancing his mortgage. The previous section showed how we create a tree based on a two-factor dynamic interest rate model that reflects conditions in the Danish MBS market well. But the stochastic optimization model itself is completely detached from the interest rate/term structure generating model used. Any two-factor interest rate model could hence be in conjunction with the Stochastic Programming model.

The objective of the model is to optimize some measure of expected payments, suitably discounted over time, and is discussed in Section 4.2. Below we discuss the constraints of the model (balance equations), whose primary purpose is to keep track of remaining principal versus payments, under consideration of possible re-balancing, and across nodes in the tree.

4.1 Balance Equations

The balance equations manage outstanding principal amounts of each potential bond (loan type), between pairs of tree nodes, (i, k), where node i is the predecessor of node k. The only node without a predecessor is the root node.

We consider the simplest case where there are no nodes are one coupon payment apart; 3 months in real life. At node (i, k) there is a finite universe of traded bonds, indexed by i, and given the term structure and the spreads we can find market prices of these. Specifically, let:

\[ P_k^i \] be the discount factor of node \( k \)'s term structure for a loan maturing at time \( t_k + \tau \), i.e., \( \tau \) years after the node time.

\[ R_k^i \] be the present value of bond \( i \)'s non-callable version. The main task here is determining

![Figure 4: The decision tree. The square boxes are where we can actively make a decision in the optimization problem (in other words the \( k \)'s that we sum over in equation (7)). At each of these nodes we have a complete term structure and a universe of MBS bonds. The bullet points are other dates at which cash-flows occur.]
the cash-flows, where we need the annuity formula saying that the installments for an n-period annuity with coupon rate \( C \) are \( (C \cdot (1 + C)^n)/(1 + C)^n - 1) \).

\( I^i_k \) be the market price of bond \( i \) at node \( k \). When \( i \) corresponds to an ARM bond \( I^i_k = B^i_k \), since ARMs aren’t callable. The prices of callable MBS are found with the spread in Section 3.2. The mortgagee can sell short this bond if its price is at or below par, say \( P \). If \( k \) already has a short position, he may cancel this at a cost of \( \min(I^i_k, P) \); he either buys the bond at the market price or he “calls”.

We need the following decision variables:

- \( x^i_k \): Outstanding principal amount of bond \( i \) at node \( k \); for a node-ancestor pair \((a,k)\), the balance equations serve to link together \( x^i_k \) and \( x^j_{k'} \).

- \( p^i_k \): Scheduled payment on principal at node \( k \). For an annuity we have installments = interest payment \( = p_k^i \cdot p_k^d \). We also need to keep track of interest payments for tax purposes since interest payments are tax-deductible in Denmark (currently at a rate of 32%).

- \( p^d_k \): Principal called, i.e., “extreme binary payment of principal” for type \( i \). When \( i \) corresponds to the non-callable ARM-bonds we have \( p^d_k = 0 \).

- \( p^s_k \): Principal sold, i.e., principal of type \( i \)-bonds sold to raise funds, \( \sum_k \sum_i p^s_k p^d_k = \text{DKK} 2,000,000 \).

- \( p^d_k \): Principal delivered, i.e., principal of type \( i \)-bonds bought back at node \( k \). We only allow “net short positions” for any type of bond; you cannot borrow huge amounts (in short term loans) with you house collateral and use the money to buy long-term bonds.

We have a balance constraint saying that for bond \( i \) the principal amounts of debt at nodes \((a,k)\) satisfy

\[ x^i_k - p^i_k - p^d_k + p^s_k = x^i_{k'} + p^d_{k'} \]

So the total payment (cash out) at node \( k \) is

\[ T^k = \sum_i \left[ p^i_k + (1 - \xi) \cdot p^d_k + p^d_k - (1 - \beta) \cdot I^i_k \cdot p^d_k + I^i_k \cdot p^d_k \right] \]

where \( \xi \) is the tax rate and \( \beta \) is the (proportional) rate of transaction costs on sales (we consider only transaction costs on sales). Proportional transaction costs are incorporated by setting the parameter \( \beta \) to the appropriate percentage cost of selling (selling) bonds.

Fixed costs can be modeled using binary variables, which would make the model substantially harder to solve, all else being equal. Notice that the transaction costs create path dependence, the costs associated with a “buy” portfolio depend on the “sell” portfolio you currently have and its market value.

As initial conditions we require that we must raise a minimum DKK 2,000,000, but as long as transactions costs are proportional we could use any amount. As terminal condition we require that everything must be paid back after at most 30 years, whatever is left after 30 years is paid off at market price.

Note and ancestor multiple coupon payment dates apart

In between the \( t_j \)’s in Figure 4 we are not allowed to reorganize our position, but we still have to keep track of the cash-flow. This is a major complication of the model’s bookkeeping which does not alter the basics of what was given above.

For callable MBS annuities: Given the holdings of bond \( i \) at \( t_j \) all ordinary after-tax payments at \( s_1, s_2, \ldots, s_m = t_{j+1} \) are known, say \( x(t) \). The time \( t_j \) forward values are \( x'(t)/P(t, t_{j+1}) \), and these we simply take appropriately into account in the “net-cash-outflow at \( t_j \)” equation (6). The remaining principal at time \( t_j \) is also calculated.

For ARMs: Not rearranging the portfolio for ARMs means keep reinvesting in the same type \( F \)-year ARM. To model this succession of equal-type bonds with differing rates, we use linear interpolation of the intermediate rates, from the \( F \)-year rate in effect at nodes \( a \) and \( k \).

4.2 Objective Functions

To get a feel for the model we initially consider a risk-neutral borrower, i.e., one that seeks to minimize expected present value of lifetime on his mortgage:

\[ \text{Minimize } z = \sum_{k=0}^{K} p_k \cdot r_k \cdot T_k^a \]

where \( k \) indexes the tree nodes, \( T_k^a \) is the total payment at node \( k \), which has probability \( p_k \), and \( r_k \) is the discount factor for payments at node \( k \). Note that the discount factor can be state dependent. But in our implementation it only depends on the time to node \( k \), so we might as well write \( r_k \), because we just use the initial yields curve to discount.

Risk-neutrality is probably not a completely realistic assumption about homeowners. Bying a house is the largest investment most people ever make, so the mortgage payments are likely to have a significant impact on their private economy and its management possibilities, something that is enhanced by a strong aversion to losing their (current) home (which is
5 Numerical Experiments

The purpose of the experiments presented here is to "sanity check" the model, i.e., verify that it behaves reasonably, and also to demonstrate its feasibility with respect to solution times for large non-trivial instances (up to 8 stages, for a total of 9941 nodes in the tree). The effect of varying the short- and long-rate volatilities, as well as their correlation, is investigated.

5.1 Base Case Models

We choose as our base case model parameters the volatilities estimated in Table 1. The initial yield curve used is estimated by the average observed rates in Figure 1: a 1-year rate of 4.4% and a 30-year rate of 6.3%. The model is set up to accommodate a number from 3 to 5 stages, with the values of \( T \) as shown in Table 3, and for the base case we select 5 stages; this captures the compromise between model accuracy and speed of solution.

We include a representative mix of ARMs and callable annuities in the bond universe. Specifically:

- ARM with refinancing frequencies of 1, 2, 3, 4 and 5 years (abbreviated ARM-n).

<table>
<thead>
<tr>
<th>Stages</th>
<th>Decision points, ( T )</th>
<th>Nodes</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>(0, 1, 5, 30)</td>
<td>48</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>(0, 1, 5, 10, 30)</td>
<td>121</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>(0, 1, 2, 5, 10, 30)</td>
<td>364</td>
<td>13</td>
</tr>
<tr>
<td>6</td>
<td>(0, 1, 2, 5, 10, 20, 30)</td>
<td>1095</td>
<td>37</td>
</tr>
<tr>
<td>7</td>
<td>(0, 1, 2, 5, 10, 20, 30)</td>
<td>3260</td>
<td>443</td>
</tr>
<tr>
<td>8</td>
<td>(0, 1, 2, 5, 10, 15, 20, 30)</td>
<td>9841</td>
<td>8321</td>
</tr>
</tbody>
</table>

Table 3: The multistage stochastic models: Number of stages, time points, nodes in the resulting tridiagonal tree, and solution times in seconds on a 800MHz Pentium computer with 128MB of RAM, using GAMS Rev 121 and CPLEX 7.0.

- Callable annuities of maturities 1, 2, and 30 years with coupon rates of 4.18, 5.10, 6.12, and 6.14% (abbreviated Fox n-C).

A proportional transaction cost of 1.5% is assumed. This rate is quite close to the actual refinancing costs for a DKK 2,000,000 loan. About 1% of this figure is actually a fixed cost (tax), but we chose to model the fixed costs in the preliminary study due to the expense of solving mixed-integer models. We would not expect this to have a noticeable effect on the model's selection of initial portfolio.

5.2 Base Case Loans

The base case results, against which we subsequently perform comparisons along different dimensions, are given in Table 4. We show the loans held at each node where refinancing occurs. Only data pertaining to the 40 nodes during the first four stages are shown. For instance, it is seen that the initial, optimal portfolio is the 2-year ARM. During the subsequent Up-scenario, the model shifts into the 30-year fixed 6% loan, under the Down-scenario into the 30-year fixed 9% loan. During the Middle scenario, the original ARM is retained. The scenarios are named after the movements of the long rate. Also, since the objective is linear and therefore models a risk-neutral investor, the model will never hold mixed loans portfolio.

The interesting feature of these results is primarily that the model utilizes both the possibility of converting up and down, and that this behavior follows, primarily, the movements of the long rate. This clearly indicates that the possibility of delivery, a.k.a. "converting up" (absent in, for instance, the U.S. market) has value and further studies will allow an
<table>
<thead>
<tr>
<th>Note</th>
<th>fi</th>
<th>Short</th>
<th>Long</th>
<th>Loan Held</th>
</tr>
</thead>
<tbody>
<tr>
<td>root</td>
<td>0</td>
<td>4.4</td>
<td>6.5</td>
<td>ARM-2</td>
</tr>
<tr>
<td>U</td>
<td>1</td>
<td>6.5</td>
<td>8.2</td>
<td>Fx(33-06)</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>4.4</td>
<td>4.8</td>
<td>Fx(33-04)</td>
</tr>
<tr>
<td>U-U</td>
<td>2</td>
<td>9.6</td>
<td>10.4</td>
<td>Fx(33-08)</td>
</tr>
<tr>
<td>M-U</td>
<td>2</td>
<td>4.4</td>
<td>8.7</td>
<td>Fx(33-06)</td>
</tr>
<tr>
<td>M-M</td>
<td>2</td>
<td>2.0</td>
<td>7.3</td>
<td>Fx(33-04)</td>
</tr>
<tr>
<td>M-D</td>
<td>2</td>
<td>3.0</td>
<td>5.1</td>
<td>Fx(33-04)</td>
</tr>
<tr>
<td>U-U-M</td>
<td>5</td>
<td>33.7</td>
<td>22.1</td>
<td>Fx(33-14)</td>
</tr>
<tr>
<td>U-U-D</td>
<td>5</td>
<td>8.7</td>
<td>16.3</td>
<td>Fx(33-16)</td>
</tr>
<tr>
<td>U-D-D</td>
<td>5</td>
<td>17.1</td>
<td>8.8</td>
<td>Fx(33-16)</td>
</tr>
<tr>
<td>M-U-M</td>
<td>5</td>
<td>8.7</td>
<td>16.3</td>
<td>Fx(33-16)</td>
</tr>
<tr>
<td>M-D-D</td>
<td>5</td>
<td>2.2</td>
<td>12.0</td>
<td>Fx(33-08)</td>
</tr>
<tr>
<td>U-D-M</td>
<td>5</td>
<td>17.1</td>
<td>8.8</td>
<td>Fx(33-16)</td>
</tr>
</tbody>
</table>

Table 4: Base case loans: Nodes where rebalancing occurs during the first 4 periods are shown.

<table>
<thead>
<tr>
<th>Stages</th>
<th>Root Portf</th>
<th>Up</th>
<th>Middle</th>
<th>Down</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>ARM-1</td>
<td>Fx(33-06)</td>
<td>Fx(33-04)</td>
<td>Fx(33-04)</td>
</tr>
<tr>
<td>4</td>
<td>ARM-1</td>
<td>Fx(33-06)</td>
<td>Fx(33-04)</td>
<td>Fx(33-04)</td>
</tr>
<tr>
<td>5</td>
<td>ARM-2</td>
<td>Fx(33-06)</td>
<td>ARM-2</td>
<td>Fx(33-04)</td>
</tr>
<tr>
<td>6</td>
<td>ARM-2</td>
<td>Fx(33-06)</td>
<td>ARM-2</td>
<td>Fx(33-04)</td>
</tr>
<tr>
<td>7</td>
<td>ARM-3</td>
<td>Fx(33-06)</td>
<td>ARM-3</td>
<td>Fx(33-04)</td>
</tr>
<tr>
<td>8</td>
<td>ARM-3</td>
<td>Fx(33-06)</td>
<td>ARM-3</td>
<td>Fx(33-04)</td>
</tr>
</tbody>
</table>

Table 5: The optimal loans held initially and at 1 year, the latter under the Up, Middle and Down stages.

Finally, we note that there were a total of 54 rebalancing opportunities during the 8-year horizon at the next stage (corresponding to 5 years). In other words, the probability that a refinance will become optional during years 2 and 5 is 2/3. This is true only if rebalancing is not allowed in-between, and point to the need for a finer time resolution, especially at the beginning of the horizon. The model’s apparent indication of a 2/3 chance that defaulted Danish mortgage-owners should refinance during any 3-year period is quite high (certainly higher than what we expected) and can be taken as indication that the financial institution may be correct in frequently suggesting refinancing of mortgages.

Solution times for the 3-8 stage models are shown in Table 3. These times are measured in seconds on an 880 MHz Pentium with 128 Mbytes of RAM and running Windows ME. The models are expressed entirely using the General Algebraic Modeling System, GAMS (Brooke, Kendrick & Meeraus (1992)), and the LP solver used was CPLEX 7.0. Standard settings for CPLEX were used throughout. The largest model contained 1,361,162 constraints and 1,649,096 variables. Of the times shown, CPLEX accounted for roughly 25%; the rest was spent by GAMS on generation and execution time. It is worthwhile to note that these stochastic programs of substantial size were solved using off-the-shelf software and hardware.

5.3 Stages and Model Results

We show in Table 5 the model’s results, as indicated by the initial and second-stage portfolio, as the number of stages is varied.

The results show a remarkable stability across the number of stages. In each case, the optimal, initial, loan is an ARM. The optimal loan in each case is clearly the 3-year...
ARM - the reason that is not chosen by the first 4 models is that these models do not have a re-balancing point that allows them to re-balance away from it after 3 years. This clearly shows a need to include proxies for multi-year successions of different ARMs over time periods where the model does not allow re-balancing, especially during longer such periods. In addition, the model always shifts to the 30-year fixed 9% when long rates increase during the first year, and to the 30-year fixed 4% when they decrease. When they stay unchanged, it keeps the initial loan (except where that loan, for reasons previously mentioned, expires).

5.4 The Risk-Averse Investor

A primary objective to the linear objective function is that it models an unlikely, risk-neutral investor. We solve in this section the 4-stage model using a non-linear utility function as the objective. More precisely, it is for this exercise assumed that the investor has a fixed budget \( B_0 \) available at each point \( \tau_i \) in the future, and wants to maximize expected utility of surplus at future times. Specifically, our objective is to:

\[
\text{Maximize } z_{d, 2} = \sum_{k=1}^{N} \beta^k \cdot \log \left( \psi \cdot (B_{n+k} - T^k) \right)
\]

This is only one of many ways to address risk-aversion.

Interestingly, this objective leads to a diversified loan portfolio. The initial portfolio is now a 10-year fixed-rate 4% loan instead of an ARM. At the second stage (1 year), the model switches into another portfolio at the Up node, this portfolio diversified between the 10-year fixed 6% (70%) and the 30-year fixed 6% (30%). At the second-stage Down node, the portfolio held again diversified, this time between the same 10-year fixed 6% (80%) and the 30-year fixed 4% (20%). Diversification continues throughout the tree at stage 3 as a portfolio is even diversified 3 ways. In fact, this leads to an interesting "policy recommendation", because the currently Danish tax-system discourages diversification, because there is (as opposed to our model) a lump-sum tax on loaning hands. The Minister of Taxation recently suggested changing this, and we see that this would benefit risk-averse houseowers.

The risk-averse behavior is here evident in two respects. First, the initial portfolio is a relatively conservative fixed-rate of a relatively short-duration loan, second, many subsequent portfolios are diversified.

For this experiment, Minos 5 was used. We did not have access to solvers capable of solving the larger, non-linear instances. Solution times were reasonable, but due to differences in solvers and computers it would not be appropriate to make precise comparisons here.

5.5 Effects of Volatilities and Correlation

It is expected that the short rate volatilities and their correlation significantly influence the optimal portfolio, in particular, whether it is optimal to use ARMs or callable fixed-rate loans. To test this hypothesis, we perform several investigations. Technically, we changed into the two-factor model suggested in Nielsen & Roen (1997) because of its added correlation flexibility, but because we still calibrate to appropriate volatilities and fixed-rate structure this has no bearing on the numerical results. We state the results in terms of intuitively understandable short-rate proportional volatility, \( \sigma_s \), long-rate proportional volatility, \( \sigma_l \), and their correlation, \( \rho \). In Table 6 we keep the short and long volatilities equal to various levels, and uncorrelated. The results show that the model as expected prefers ARMs at low volatilities, and shifts into fixed-rate loans at high volatilities. The changeover occurs between 20% and 35%. To check how far this changeover point is from the base case volatilities of \( \sigma_s = 3.2 \% \) and \( \sigma_l = 22 \% \), we "shift" these volatilities by the same number, the optimal portfolio shifts into the fixed-rate at the point \( \sigma_s = 42 \% \) and \( \sigma_l = 32 \% \).

Further testing showed that the decision between ARM and fixed is determined almost exclusively by the long volatility. In fact, changing the short volatility alone did not cause a switch to fixed. Also, the correlation in this case did not influence the choice at all.

Finally, the performing non-parallel shifts to the term structure (relative to the base case) shows that when the short end drops significantly, and the long end increases, there is a switch into fixed.

This is altogether solid evidence that at the present levels of long rate volatilities it is
certainly optimal to use ARM for the risk-neutral investor.

5.6 Effects of Transactions Costs

The level of proportional transactions costs was increased from its base case value of 1.5%. At 10% there is no rebalancing until stage 4 (5 years), where it occurs with probability 37%. At 50% the probability drops to 15%, and at 50% to 4%. The effect of varying the transaction cost is as expected, with less rebalancing the higher the transaction cost is. Although the model seems somewhat more eager to rebalance than we would have expected (as previously noted), there is no doubt in the significant value to the mortgage of having these options.

6 Conclusions

We have presented a two-factor model for the Danish MBS market and used stochastic programming techniques to investigate the optimal (long-) portfolio problem for an individual investor. We find that risk-neutral investors should adjust their portfolio frequently and take advantage of the delivery option built into fixed rate securities. In other words we do find support for the advice given by financial institutions. Mortgages that are (liquidity) risk-neutral should use diversified portfolios of callable fixed rate securities, but also rebalancing quite frequently. The predominant factor for optimality is the long rate volatility; short rate volatilities and correlation appear to have little significance (meaning: if you get the long rate volatility right you’re OK).

There are several interesting topics for future research: Better incorporation of institutional and empirical features of the MBS market. Since the path-dependence from transaction costs is ‘of a local nature’ it may be possible to use an – at least approximate – Markovian formulation. Another way of introducing risk-aversion would be to make the discount factor state-dependent and negatively correlated with interest rates; this means that liquidity is retained. From a more practical point of view, we would like to make a detailed investigation of what causes shifts from ARM to fixed rate loans (and vice versa).

References
