Multidimensional Ito

Look at an \( n \)-dimensional Ito-process \( X \) driven by \( d \) independent Brownian motions,

\[
dX_i(t) = \mu_i(t)dt + \sum_{j=1}^{d} \sigma_{i,j}(t)dW_j(t),
\]

where \( \mu_i, \sigma_{i,j} \) are general adapted processes.

Put \( Y(t) = f(X(t)) \), where \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is a function that is \( C^2 \) in all arguments.
Important corollary

Looking at \( f(x, y) = xy \) gives us

\[
d(XY) = YdX + XdY + dXdY.
\]

This is called “the Itô product rule” (different from the “usual” one \((fg)' = f'g + g'f\)) or “stochastic integration by parts”.

It is likely to cover 90% of your applications of the multidimensional Itô formula.

Then the dynamics of \( Y \) are

\[
dY(t) = \sum_{i=1}^{n} f_i(X(t))dX_i(t) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f_{i,j}dX_i(t) \cdot dX_j(t), \tag{1}
\]

where \( f_i, f_{i,j} \) denote partial derivatives and \( dX_i \cdot dX_j \) is calculated with the formal rules (“box calculus”)

\[
\begin{align*}
&W_i^2 = dt \\
&W_i \cdot W_j = 0 \quad \text{for } i \neq j \\
&dt^2 = 0 \\
&dt \cdot W_i = 0
\end{align*}
\]
Rest of Chapter 4: Things the enemy will do to confuse you

Calculate $dX_i \cdot dX_j$ more explicitly.

Write the double summation with matrix operations.

Introduce explicit $t$-dependence.

Work with correlated Brownian motions.