EXERCISE 1; STOCHASTIC CALCULUS

In this exercise $W$ denotes a 1-dimensional (standard) Brownian motion.

1a [10%] Show that the process defined by

$$X(t) = \left(1 + \frac{1}{3}W(t)\right)^3$$

solves the stochastic differential equation (SDE)

$$dX(t) = \frac{1}{3}X^{1/3}(t)dt + X^{2/3}(t)dW(t), \quad X(0) = 1.$$ 

Show that the process defined by

$$Y(t) = \cos(W(t))$$

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solves SDE
\[ dY(t) = -\frac{1}{2} Y(t) dt - \sqrt{1 - Y^2(t)} dW(t), \; Y(0) = 1. \]

\[ 1b[10\%] \]
Consider the Cox/Ingersoll/Ross SDE
\[ dX(t) = \kappa(\theta - X(t)) dt + \sigma \sqrt{X(t)} dW(t), \; X(0) = x. \]

Show that
\[ m(t) := \mathbb{E}(X(t)) = X(0) e^{-\kappa t} + \theta (1 - e^{-\kappa t}). \]

Hint: Write the SDE on integral form, take mean, interchange (don’t worry about technicalities; it is allowed), and observe that the \( m \)-function solves an ordinary differential equation (ODE).

Derive an ODE for the second moment, \( h(t) := \mathbb{E}(X^2(t)) \).

Hint: Use Ito on \( X^2 \) and do as in the first part of the question.

Remark: You are not asked to solve the second moment ODE. If you (or MAPLE) do, and manipulate the results a little, you’ll find that
\[ \text{var}(X(t)) = X(0) \frac{\sigma^2}{\kappa} (e^{-\kappa t} - e^{-2\kappa t}) + \theta \frac{\sigma^2}{2\kappa} (1 - e^{-\kappa t})^2. \]
Because \( X \) is time-homogeneous Markov process conditional moments are found by shifting the formulas/equations.

**EXERCISE 2: BLACK/SCHOLES OPTION PRICING** (20%) 
In this exercise we consider the base-case Black/Scholes model.

Let (the power) \( p \in \mathbb{R} \), (the strike) \( K \in \mathbb{R}_+ \), and (the expiry date) \( T \in \mathbb{R}_+ \) be given and consider the powered call, which is a simple contingent claim that that pays \( (S^p(T) - K)^+ \) at time \( T \).

Derive a closed-form expression for the arbitrage-free price of the powered call.

Hint: “Brute force” works, but using Ito on \( S^p \) and remembering how the Black/Scholes formula looks for dividend-paying stocks can save you calculations.

Assume \( T = 1, \; K = 10, \; p = 1/2, \; S(0) = 100, \; r = 0.02 \) and \( \sigma = 0.2 \). Calculate (numbers, please) the time-0 price of the powered call.
Explain how the powered call can be dynamically hedged using the stock and the bank-account. How many units of stock must you hold?

EXERCISE 3: OPTION PRICING WITH A TWIST (20%)

In this exercise we consider a general arbitrage-free stock price model in a world where there exists a bank-account on which the interest rate is 0, i.e.

\[ dS(t) = \sigma(t)S(t)dW^Q(t), \]

where \( \sigma \) is an arbitrary (up to regularity conditions, that you needn’t worry about) stochastic process.

Consider a simple contingent claim on \( S \) with pay-off function \( g \), i.e. it pays \( g(S(T)) \) to the holder at time \( T \).

Let \( \tilde{\sigma} \in \mathbb{R} \) be given and define the function \( F : [0; T] \times \mathbb{R} \rightarrow \mathbb{R} \) as the solution to the (familiar) partial differential equation (subscripts denote differentiation)

\[ F_t(t, x) + \frac{1}{2} \tilde{\sigma}^2 x^2 F_{xx}(t, x) = 0 \text{ for } t < T, \quad F(T, x) = g(x). \]

Consider, finally, a trading strategy \( h \) that holds \( h_1(t) = F_x(t, S(t)) \) units of the stock and \( h_2(t) = F(t, S(t)) - S(t)F_x(t, S(t)) \) units of the bank account at time \( t \).

Show that this trading strategy replicates the pay-off of the \( g \)-claim, i.e. that its value process, say \( V^h \), satisfies \( V^h(T) = g(S(T)) \).

Show that the self-financing condition for this trading strategy boils down to the equation

\[ \frac{1}{2} (\sigma^2(t) - \tilde{\sigma}^2)S^2(t)F_{xx}(t, S(t)) = 0 \] (1)

holding (almost everywhere, in an appropriate sense, that you needn’t worry about).

Argue that “usual results” are obtained when \( \sigma(t) = \tilde{\sigma} \), and that in general the \( h \)-strategy can be interpreted as “trying to replicate as if it were the Black/Scholes model”.

Remark: The result in (1) may alternatively be formulated by saying that \( h \) has an extra financing need of

\[ \frac{1}{2} \int_0^T (\sigma^2(t) - \tilde{\sigma}^2)S^2(t)F_{xx}(t, S(t))dt. \]
and is sometimes called the “1st fundamental theorem of derivative trading”. It has consequences for hedging in misspecified models: If we consider a convex claim (in the sense that $F_{xx} > 0$) and if there is an upper bound on the $\sigma$-process, then a super-replicating strategy is achieved by “Black/Scholes $\Delta$-hedging with the upper bound”.

**EXERCISE 4: LONG RATES (20%)**

This exercise is very much inspired by Björk’s exercise 22.7.

In term structure modelling it is often considered a reasonable request that the forward rate curve has a horizontal asymptote, i.e. $\lim_{T \to \infty} f(t, T)$ exists for all $t$.

Remark: Björk says that “obviously the limit will depend on $r(t)$ and $t$”. To me seems more obvious that the limit – if it exists – does not depend on $r(t)$ and $t$. Neither is true, but it can be shown that if the process $f^\infty(t) := \lim_{T \to \infty} f(t, T)$ is well-defined, then it is increasing. And that there are models – fairly strange ones, though – where the limit depends non-trivially on $r(t)$.

4a [10%]

**Show that** in the Vasicek model, the request is fulfilled, i.e. there is indeed a horizontal asymptote. **Find it.**

4b [10%]

Consider a Ho/Lee-model calibrated (via $\theta$ as usual) to an initially observed forward rate curve, say $f^*(0, T)$.

**Show that**

$$f(t, T) = f^*(0, T) + \sigma^2 t \left( T - \frac{1}{2} t \right) + \sigma W^Q(t) \quad \text{for all } T \geq t \geq 0.$$

Use this to **show that** even if the initial forward rate curve has a horizontal asymptote, none of the forward rate curves that the model subsequently generates have this property.

Remark: The result in question 4b is an example of a modelling inconsistency across the calendar time and maturity time domains. It is a subject that Björk (with co-authors) has studied a great deal. A strange result (obtained with doses of stochastic geometry and Lie-brackets) is that the Nelson/Siegel-curve (which you may or may not have heard of) is inconsistent with any finite dimensional (which you haven’t heard of anything that isn’t) arbitrage-free term structure model.
EXERCISE 5: OPTIONS ON COUPON BONDS (20%) 

In this exercise we consider the Vasicek model,
\[ dr(t) = \kappa(\theta - r(t))dt + \sigma dW^Q(t), \]
assume (as always) that $\kappa > 0$, and let $A$ and $B$ denote the functions such that $P(t, T) = \exp(A(t, T) - B(t, T)r(t))$.

We now consider a coupon bond that makes deterministic positive payments $\alpha_1, \ldots, \alpha_N$ at dates $T_1, \ldots, T_N$. Clearly the price of this coupon bond is
\[ \pi^C(t) = \sum_{i | T_i > t} \alpha_i P(t, T_i). \]

(It is strict inequality, ”>”, to keep in line with prices always being ex-dividend.) The last ingredient we need is a (positive) strike-$K$, expiry-$T$ European call-option on the coupon bond.

**Show that** there exists a unique $r^* \in \mathbb{R}$ such that $\pi^C(T) \geq K$ if and only if $r(T) \leq r^*$. Hint: Use the Vasicek assumption and ask yourself “when is $\pi^C(T) = K$?”

Define the adjusted strikes via
\[ K_i = \exp(A(T, T_i) - B(T, T_i)r^*). \]

**Show that** the pay-off of the call can be written as
\[ (\pi^C(T) - K)^+ = \sum_{i | T_i > T} \alpha_i (P(T, T_i) - K_i)^+. \]

Hint: Two things can happen to the right hand side. Investigate these separately.

**Explain how** (given results known from for instance Björk) this leads to a closed-form (up to knowledge of $r^*$) expression for the price of the call on the coupon bond.

Assume
- $\theta = 0.05$, $\kappa = 0.2$, $\sigma = 0.01$, $r_0 = 0.05$
- $T = 1$, $N = 2$, $T_i = i + 1$ and $\alpha_i = 1$ for $i = 1, 2$, $K = 1.856$.

**Calculate** the time-0 price of the call. (Numbers, please; this will involve solving an equation numerically.)

Remark: The exact same trick (due to Farshid Jamshidian) reduces a call on a portfolio in the Cox/Ingersoll/Ross to a portfolio of ZCB calls.
Unfortunately, the trick does not work in multidimensional models.