Mathematical Finance, Fall 2004, note 4 (week 39)

Past lectures
Monday September 20: The rest of Björk’s chapter 5.
Wednesday September 22: Björk’s chapter 6 where we come back to finance. The first few pages of chapter 7. We looked at some actual trading strategies; the exercises will look closer. There’s data and R-code on the homepage.

Coming attractions
Monday September 27, Wednesday September 29: Onwards in chapter 7. Arbitrages and how to spot them. The Black/Scholes equation and formula.
The exercises for Tuesday September 28 are given below. They should more than suffice for two hours of entertainment.

Kindly,
Rolf

Exercises for week 40 (Tuesday September 28)

First some things you shouldn’t spend time on in class: Björk’s exercise 4.8 (left over from last week), Björk’s exercises 5.2 and 5.3, (that show that – once we learn about matrix exponentials – things work the same in the multidimensional case as in the 1-dimensional one from exercise 4.2 below), and Björk’s exercise 6.1 (the self-financing condition with dividends; relevant but boring and we will not need it for a while; exercises 4.4 and 4.5 below are much more important for understanding self-financing).

Exercise 4.1: More Geometric Brownian motion
Consider a Geometric Brownian motion as seen in Section 5.2,

\[ dX(t) = \alpha X(t)dt + \sigma X(t)dW(t), \]
where \( X(0) = x_0 > 0 \).

Find \( \text{var}(X(t)) \), and explain why there’s no need to waste brain RAM remembering this. While you’re at it: Solve Björk’s exercise 5.5

Does \( X \) have independent increments?

Hint: Take \( t > s \), calculate \( \mathbb{E}(X(s)(X(t) - X(s))) \) (use iterated expectations/the tower law), and compare it to \( \mathbb{E}(X(s))\mathbb{E}(X(t) - X(s)) \).

Assume now that \( 0 < \alpha < \sigma^2/2 \). Show that

\[
\mathbb{E}(X(t)) \to \infty \text{ for } t \to \infty,
\]

while on the other hand

\[
X(t) \to 0 \text{ almost surely when } t \to \infty.
\]

Hint: Look at \( (\alpha - \sigma^2/2)t + \sigma W(t) = t((\alpha - \sigma^2/2) + \sigma W(t)/t) \) and use the law of large numbers. (Strictly speaking, this leaves a little gap, because time is continuous. Ignore it, or see how to solve it on Note 1.)

**Exercise 4.2: The affine SDE/The Ornstein/Uhlenbeck process**

Solve Björk’s exercise 5.1 with the slightly more complicated \( X \)-process:

\[
dX(t) = (\gamma + \alpha X(t))dt + \sigma dW(t).
\]

The hint still works, although “look at \( e^{-\alpha t}X(t) \)” would be shorter.

Determine the mean function, \( m(t) = \mathbb{E}(X(t)) \), and the covariance function \( c(s, t) = \text{cov}(X(s), X(t)) \).

Hint: Take \( t > s \), use something like Björk’s equation (5.20) (twice) to express \( X(t) \) in terms of \( X(s) \) and a \( dW \)-integral of a deterministic function over \([s; t]\), and use iterated expectations to calculate \( \mathbb{E}(X(t)X(s)) \). If you are meticulous, then repeat the calculation for \( t \leq s \).

So does \( X \) have independent increments?

**Exercise 4.3: Feynman/Kac with an extra term**

First, do Björk’s exercise 5.10. Then, use it to solve his exercise 5.11, which is pretty much like 5.9, that I did at the lectures.

**Exercise 4.4: Trading strategies**

At the lectures on Wednesday September 22 we looked at some specific trading strategies, and how they worked with real data. You can find the R-code and the actual data (that are not really important to this exercise) on the homepage. Recall that 3 different types of investors were considered: The
buy-and-hold, the contrarian, and “Mr. 50%” who holds a constant proportion of his portfolio value in the stock.

Show that the buy-and-hold investor’s portfolio is indeed self-financing.

Show that the contrarian investor’s portfolio is self-financing. Why is he called “contrarian”?

Show that ‘Mr. 50%’’s portfolio (i) does what it is said to do, (ii) is self-financing, (iii) could be called (slightly) contrarian.

Exercise 4.5: Cheating strategies
Consider trading like this:

```R
V[1]<-100
for(i in 1:(nobs-1)){
  if (KFX[i+1]-KFX[i] > 0){
    hKFX[i]<-V[i]/KFX[i]
    hBANK[i]<-0
  }
  if(KFX[i+1]-KFX[i]<=0){
    hKFX[i]<-0
    hBANK[i]<-V[i]/bankaccount[i]
  }
  V[i+1]<-hKFX[i]*KFX[i+1] +hBANK[i]*bankaccount[i+1]
}
```

(There’s code on the homepage.) How does the value process for this strategy look? And which part of our usual and sensible sensible assumptions is violated?

Would changing the logical conditions above to something like

```R
if (bankaccount[i+1]-bankaccount[i] > 0)
```

lead to an allowable trading strategy?

Consider the trading strategy defined by

```R
V[1]<-100
hKFX[1]<-0.5*V[1]/KFX[1]
hBANK[1]<-0.5*V[1]/bankaccount[1]
```
for(i in 2:nobs){
  V[i]<-hKFX[i-1]*KFX[i]+hBANK[i-1]*bankaccount[i]
  hKFX[i]<-0.5*V[i]/KFX[i]
  hBANK[i]<-hBANK[i-1]
}

Is it self-financing?