Mathematical Finance, Fall 2004, note 2 (week 37)

Past lectures
Monday September 6: The rest of Björk’s sections 4.1-4 + Proposition B.37 from Appendix B.

Wednesday September 8: Simulation via the Euler scheme (see slides or code). Ito’s formula and a proof along the lines of Remark 4.5.2.

Coming lectures
Monday September 13 and the following lectures: A quick & dirty proof of the Ito formula. Applications. Multi-dimensional Ito.
Questions for the exercises on Tuesday September 14 are given below.

Kindly,
Rolf

Exercises for Week 38 (Tuesday September 14)
Exercise 2.3 closely resembles something I’ll do at the lectures on Monday September 13, so waiting until after these, will probably save you time.

Exercise 2.1: New martingales from Brownian motion
Let $W$ be standard Brownian motion. Show that the following processes are martingales:

- $M_1(t) = W(t)$
- $M_2(t) = W^2(t) - t$
- $M_3(t) = \frac{W^3(t)}{3} - tW(t)$
You should be able to prove this both from “first principles” and by using the Ito formula.
Show that these martingales have the orthogonality property $\mathbb{E}(M_i(t)M_j(t)) = 0$.

**Exercise 2.2: Routine with Ito**
Björk Exercises 4.1 and 4.2.

**Exercise 2.3: An important result**
Solve Björk’s Exercise 4.3.
This is a very important result and the conclusion hinges strongly on $\sigma$ being deterministic. (So you should be very clear in your argumentation about where it is used.) To do things properly, split into real and imaginary parts, i.e. $e^{iuX(t)} = \cos(uX(t)) + isin(uX(t))$. Then recall that the characteristic function determines the distribution, i.e. if two random variables have the same characteristic function, they have the same distribution. If you haven’t taken something like Stat 1B you may rightfully claim that you don’t know that. Then you probably don’t known either that the characteristic function of $X \sim N(\mu, \sigma^2)$ is $\phi_X(t) := \mathbb{E}(\exp(itX)) = \exp(it\mu - t^2\sigma^2/2)$, which at least looks plausible. (Why?)

**Remark I:** It is really not necessary to draw in complex numbers. You can reach the same conclusion by doing calculations on the moment generating function, $m(u) = \mathbb{E}(\exp(uX(t)))$. Both determine the distribution. There is a technical reason Björk looks at the characteristic function: Since $e^{iuX(t)}$ is on the unit circle in the complex plane, so the expectation is always finite. The moment generating function can (easily) be infinite even if $u$ is close to 0, in which it doesn’t determine anything.

**Remark II:** In fact $X$ is a Gaussian process with independent increments, i.e. for any $n$ and ordered $t_i$’s, the vector $(X(t_1), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1}))$ is normally distributed with independent coordinates. If you have taken Stat 1B you can actually prove it with characteristic functions directly from the definition of the stochastic integral. Another way is to show that $X(t), X(s)$ and $X(t) - X(s)$ are all normally distributed, which follows by the previous arguments when using that $X(t) - X(s) = \int_s^t \sigma(u) dW(u)$). The “independent increments”-conclusion also hinges on $\sigma$ being deterministic. For general Ito-processes it is not true that $X(t) - X(s)$ is independent of $X(s)$, although it is tempting to say so. Just think of Geometric Brownian motion as a counterexample.
Exercise 2.4: Simulation & the extra “$\frac{1}{2} \int x \sigma dt$”-term

Consider the stochastic differential equation (SDE)

$$dY(t) = \sigma Y(t) dW(t),$$  \hspace{1cm} (1)

where $W$ is a standard one-dimensional Brownian motion. Assume that $Y(0) = 1$ and for the numerical exercise let $\sigma = 0.25$. If we did not know of the strange properties of Brownian sample paths and used the theory of ordinary differential equations, we might think that the solution to (1) is given by

$$Y(t) = \exp(\sigma W(t)).$$  \hspace{1cm} (2)

1. Show that this solution would indeed hold if $W$ were differentiable and we had $dW(t) = W'(t) dt$.

Using your favorite computer software (R is good, but Excel is fine too) do the following simulation exercise. (In all of the following, think of the time unit of $t$ as years.)

1. Simulate a path of a Brownian motion over 10 years with an observation point at each $\frac{1}{100}$ years.

2. Using this path, simulate the solution $Y(t)$ to the SDE (1) over 10 years, subdividing each year into 100 equal time length periods, letting $Y(0) = 1$ and letting

$$Y(t_{i+1}) = Y(t_i) + \sigma Y(t_i)(W(t_{i+1}) - W(t_i)).$$

Use the Euler scheme, in other words.

3. Use the same path as above to simulate the two processes below in the interval 0 to 10 years (and using the same subdivision)

$$Y_1(t) = \exp\left(\sigma W(t) - \frac{1}{2} \sigma^2 t\right)$$ \hspace{1cm} (3)

$$Y_2(t) = \exp(\sigma W(t))$$ \hspace{1cm} (4)

Which of these two solutions best approximates the answer from 2?

4. If you are excited, repeat the simulations above 100 (or 1000) times. Compute the expectations of $Y(T)$ as found in (2) and $Y_1(T), Y_2(T)$ as found in 3 by taking an average over the 100 (or 1000) values found in each simulation. (This is an example of Monte Carlo simulation.)

If presenting the answer in class, make sure to bring overhead transparencies.