Mathematical Finance, Fall 2003, Week 49

Remember the “info-meeting” for mat/øk’er on Tuesday December 9, at 16.01 in Auditorium 2. Details on the homepage.

Past lectures
Monday December 1: Chapter 17 except Section 17.4.4

Wednesday December 3: The rest of Chapter 17. Chapter 19 up to Theorem 19.8
Change of numeraire is really just a computational trick; a dimensionality reduction (like “similarity reduction” from Chapter 9). But it has proven helpful in model building, too. The idea is just that sometimes it’s convenient to measure value relative to something other than the bank-account β.

My way of arriving at/proving Theorem 19.8 is different to Björk’s. It goes like this (there is yet another way in a note I’ve posted on the homepage): Suppose $S$ is a positive price process (the stock’s, for instance) and let $\pi_i$ denote prices of traded assets (so $S$ is itself one of the $\pi_i$’s). Now we get the odd idea of looking at/for a probability measure $Q^S$ such the $\pi_i/S$ is a $Q^S$-martingale for all $i$. To put some structure on things:

$$d\pi_i = \mu_i^Q \pi_i dt + \pi_i \sigma_i^\top dW^Q,$$

where $W^Q$ is some $Q^S$-BM. (When referring to $S$ we use subscript $S$ for clarity). What can we say about $\mu_i^Q$? Using Itô (2-dimensional, see exercises if need be) on $\pi_i/S$ and the assumed martingale property (so drift = 0) we get that:

$$\forall i : \mu_i^Q S - \mu_S^Q + \sigma_S^\top (\sigma_S - \mu) = 0$$

Using this for the bank-account (where $\mu_i^Q = r$ and $\sigma_i = 0$) we get

$$\mu_S^Q = r + \sigma_S^\top \sigma_S$$

and from that

$$\forall i : \mu_i^Q = r + \sigma_i^\top \sigma_S.$$
We then immediately get Björk’s Theorem 19.8.
Further, Girsanov’s theorem tells us that the connection between the \( Q^S \)-BM and the “usual \( Q \)-BM is
\[
dW^{Q^S} = dW^Q - \sigma_S dt.
\]
In fact Girsanov’s theorem explicitly characterizes the change of measure; see exercises.
I then used the results to derive the Black/Scholes formula without writing any \( f \)-signs.

**Coming lectures**
Monday December 8, Wednesday December 10: More on change of numeraire.
The main result is Proposition 19.16 that gives an explicit formula for call options on zero coupon bonds in the (calibrated) Vasicek model. I’ll probably go straight for that.

**Exercises for Week 51 (Wednesday December 10)**
Skip the un-done PhD-course exercise from last time.
Fill out course evaluation forms. (Farhang: I’ll mail the suitably adjusted version.)

**Exercise 13.1:** Exercise 3 from the 2002-exam.
(Not a big hit among the students.) The trick is to note that \( y(t) = \alpha r(t) \) is also process of Vasicek-type and then use then known closed-form expression for ZCB prices in Proposition 17.3.

**Exercise 13.2:** Björk’s Exercise 17.7

**Exercise 13.3:** Björk’s Exercise 17.8

**Exercise 13.4:** Numeraire details/leftovers
Suppose
\[
dX = X \mu_X dt + X \sigma_X^\top dW \quad \text{and} \quad dY = Y \mu_Y dt + Y \sigma_Y^\top dW,
\]
where \( X \) and \( Y \) are 1-dimensional, but \( \sigma_X, \sigma_Y \) and \( W \) are \( d \)-dimensional.
Verify (by careful inspection of an appropriate Ito formula) that
\[
d(XY) = XY((\mu_X + \mu_Y + \sigma_X^\top \sigma_Y) dt + (\sigma_X + \sigma_Y)^\top dW).
\]
Verify (by close inspection of Girsanov’s theorem) the claims about the connection between the \( Q^S\)-BM and \( Q\)-BM. Show that the Radon/Nikodym derivative (on \( \mathcal{F}_T \)) is
\[
\frac{dQ^S}{dQ} = \frac{S(T)}{\beta(T)S(0)}.
\]

Hint: Use the general statement from Girsanov; equation (8.77) in the handouts. What is \( u \)? What is \( d(S/\beta) \)? Recognize.

**Exercise 13.5: Exercise 2 from the 1999-exam.**
(In Danish. Sorry.) Question 2a is something you should be able to do fairly easily (you have done it before). Question 2b deals with change of numeraire, and that should be your main focus in class. (But in an exam-situation, “10% means 10%”, and you should be sure to pick up the easy points.) Question 2c is about “seeing the structure”. Such questions are not uncommon.

**Exercise 13.6: Totally unrelated question**
The other day I read this on an Internet finance-site:

**Topic Title: NORMSINV(RAND()) ... not so random?**
Fri Nov 21, 03 06:05 PM

I generated 1000 random numbers in excel using NORMSINV(RAND()). Then I put them in Eviews, looked at the correlogram and it was all nice. But now, if I look at the correlogram for the first differences, the first autocorrelation is -0.52. Is it a flaw in the random number generation in excel?

Comment/answer.

Kindly,

Rolf