Past lectures

Monday November 10: Rest of Chapter 12 and all of Chapter 13

Section 12.3 is a study of domestic & foreign market prices of risk (that we really don’t need for option pricing in the complete market). It’s shown that the difference between domestic & foreign market prices must be the exchange rate volatility. This means that we cannot have risk-neutrality (market prices of risk=0) both “home & away”, where risk-neutrality is taken relative to the local bank-account. This, called Siegel’s paradox, may seem surprising, but it’s really just a “Jensen-inequality-thing” (the exchange rate seen “from the other side” is 1/X).

Chapter 13 is about various types of barrier options. These are options whose pay-offs depend on the stock-price having or not having hit some level during the whole life of the option. We didn’t go into specific formulas, but there is a clear (at least in my head) layer-structure when it comes to pricing barrier options. It goes something like this:

1. Our martingale-pricing has no problem with path-dependence, it’s still that case that \( \text{price}(t) = e^{-r(T-t)}E^Q(\text{pay-off}) \). So we could just simulate enough paths of the stock-price (whose \( Q \)-dynamics we know perfectly well) and rely in the law of large numbers to give us the price. Problem solved, I

2. The pay-off a barrier option can usually be expressed (solely) in terms of \( S(T) \) and the maximum \( M(T) = \max_{u \leq T} S(u) \). And it just so happens that for a Brownian motion \( B \), the joint distribution of \( B(t) \) and \( B^*(t) = \max_{u \leq t} B(u) \) can be determined & has a simple form. \((B, B^*)\) clearly (?) never gets outside the set \( D = \{(u, v)| u \in \mathbb{R}, v \geq u^+\}\) and with \( \Phi \) denoting the standard normal distribution function we have

\[
P(B(t) < u, B^*(t) < v) = \Phi\left(\frac{u}{\sqrt{t}}\right) - \Phi\left(\frac{u - 2v}{\sqrt{t}}\right) \text{ for all } (u, v) \in D.
\]

With this result you can do amazing things. A tiny one is to note that the barrier option pricing problem can (since a Geometric Brownian motion can be expressed in terms of a standard BM) be reduced to the calculation of a two-dimensional ordinary integral of a known function, something that there are powerful numerical methods for. Problem solved, II
3. The fact that it’s “only” $\Phi$ that pops up in the distribution function above gives us some hope that it may be possible to find closed-form solutions; perhaps even Black-Scholes-like ones. This turns out to be true – in fact Björk has some very elegant theorems (Theorem 13.8, for instance) expressing prices of barrier option prices directly in terms of the price-function for non-barrier options. *Problem solved, III*

4. The fact that, say, the knock-out call-option price is a non-too-complicated function of standard options suggest that a hedging strategy for barrier options is most conveniently made with these standard options. This turns out to be true – in fact using a strike/stock-symmetry a static hedge can be constructed; i.e. we only have to trade a time 0 (and liquidate the position when then barrier is hit). This is good thing because if we try the hedge dynamically with the stock alone (we know it’s theoretically possible) things become nasty when the barrier is approached. *Problem solved, IV*

Working through the points above would make a nice bachelor-project, or “fagprojekt”, or thesis.

**Wednesday November 12: Thoughts on the Black/Scholes/Merton formula. First part (pp 135-140) of Chapter 10.**

See slides with thoughts/ideas on the homepage.

**Coming lectures**

Monday November 17, Wednesday November 19: The rest of Chapter 10. Then on to Chapter 15 with stochastic interest rate models: First a lot of definitions, then a technical but very useful result (Proposition 15.5). It takes some time before we get from the ”abstract nonsense” to the ”concrete models”. But it’s time well spend.

**Exercises for Week 47 (Wednesday November 19)**

**Exercise 10.1: Log-optimal Portfolios**

Last year David Lando made an exercise about so-called log-optimal portfolios. Actually, he made two. You can find them on the homepage. This first version was the full time-inhomogeneous, $m$-dimensional case. That proved too hard for most (all) people. So he made a more spelled-out recipe. Start by solving that (from question 9 & onwards; use questions 1-8 to check yourself.) Go to the full version as you see fit. (Go to Exercise 10.2 first.)
Exercise 10.2: Chapter 10 & stochastic volatility models.

A stochastic volatility model for a stock price, $S$, could look like this

$$
dS(t) = \beta S(t) + \sqrt{V(t)}S(t)dW_1(t),
$$

$$
dV(t) = \xi V(t)dt + \gamma V(t)dW_2(t),
$$

where $W_1$ and $W_2$ are independent Brownian motions, and we think of $\xi$ and $\gamma$ as constants. Argue that this puts us in the realm of Björk’s Chapter 10. Think about the following:

- In the sense of Assumption 10.3.1: What should the $X$-variables be? What are the $\mu$- and $\delta$-functions?
- What is the system of equations that the $\lambda$-vector(process) must solve? In particular & in the sense of equations (10.14-15): What is the first entry of the $\alpha$-vector? What is the first row of the $\sigma$-matrix?
- Can we tell what $\lambda_1$ is? What about $\lambda_2$?

Show that under an equivalent martingale measure we have

$$
dS = rS(t)dt + \sqrt{V(t)}S(t)dW^Q_1(t)
$$

and

$$
dV(t) = (\xi - \gamma \lambda_2(t, S(t), V(t)))V(t)dt + \gamma V(t)dW^Q_2(t),
$$

and argue that if we assume that $\lambda_2$ is an honest-to-God constant (which we could call the volatility risk-premium), then the change of measure can be subsumed (“soaked up”) by a change of parameter.

Suppose the dynamics of the volatility is changed to

$$
dV(t) = \xi V(t)dt + \theta \rho V(t)dW_1(t) + \gamma \sqrt{1 - \rho^2}V(t)dW_2(t),
$$

How would you interpret the parameter $\rho$? How does this change the analysis of the “independent case”; what is now $dS$ and $dV$ under an equivalent martingale measure? To ensure that the $P$ to $Q$ change can be “soaked up” by a parameter change, what assumptions need to be made about the functional form of $\lambda_2$? What if $\lambda_1 = \lambda_2 = 0$? What if $\lambda_2 = 0$ but $\lambda_1 \neq 0$?

Kindly,

Rolf