Past lectures
Monday November 3: “So far Summary”
See the slides on the homepage.
We looked at the instantaneous $P$-expected rate of return of a call-option. When $\text{Call}(t) = F(t, S(t))$ then by Ito we have

$$d\text{Call}(t) = (F_t + \mu S(t)F_S + \frac{1}{2}\sigma^2 S^2(t)F_{SS})dt + \sigma S(t)F_SdW_P.$$ 

Now use that $F$ solves the B/S PDE,

$$F_t + rSF_s + \frac{1}{2}\sigma^2 S^2 F_{SS} = rF,$$

to write this as

$$d\text{Call}(t) = \text{Call}(t) \left( \left( \frac{(\mu - r)S(t)F_S}{F} + r \right) dt + \frac{\sigma S(t)F_S}{F}dW_P \right).$$

Therefore the instantaneous $P$-expected excess rate of return satisfies the CAPM-like equation

$$\mu_{\text{Call}} - r = \frac{S(t)F_S}{F}(\mu - r).$$

Conceivably $\mu > r$. For the B/S call-option case we have $F_S = \Phi(d_1)$ and therefore

$$\frac{S(t)F_S}{F} = \frac{S(t)\Phi(d_1)}{S(t)\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)} > 1.$$

This gives some economic explanation why options can be interesting to investors.

Wednesday November 3: More Summary & Björk 12.1-2
As an aside, I mentioned two things that are beyond the scope of this course, but interesting to look at: Portfolio choice with options & the so-called equity premium puzzle.
This we looked at currency models. When you see an exchange rate, you should not be afraid to ask “is that the number of EUROS I have to pay for 1 $, or the number of $ I get for 1 EURO?” After that, the all-important idea is to transform everything to the same currency.

**Coming lectures**
Monday November 10, Wednesday November 12: The rest of Chapter 12. Something about Chapter 13 on barrier options. And the back to Chapter 10.

**Exercises for Week 46 (Wednesday November 12)**

**Exercise 9.1: Problem 1 from the 1999 exam**
(In Danish; sorry Francesco & Christian.) The last result is general, i.e. it does not hinge a Black/Scholes model assumption. It was first derived/noted in Breeden & Litzenberger (1978), “Prices of State-Contingent Claims Implicit in Option Prices”, *Journal of Business*, vol. 51, pp 621-651. It gives us, at least in theory, a way to determine the $Q$-density of the stock price. In practice, too few strikes are available for “brute force” double numerical differentiation to give good results. Smoothing is needed – so that’s what people do. There is a great deal of literature about this.

**Exercise 9.2: Problem 1 from the 2001 exam**
Chances are, there will be something like this (i.e. stochastic calculus with no “finance disguise”) on the exam in January.

**Exercise 9.3: An Extendible Option**
Consider the base-case lognormal Black-Scholes model, i.e.

\[
\begin{align*}
    dS(t) &= \alpha S(t)dt + \sigma S(t)dW^P(t), \\
    d\beta(t) &= r\beta(t)dt.
\end{align*}
\]

Imagine that the bank where you work has developed a new product: The **extendible call-option**. This is an option where the buyer can do one of two things at maturity date $T_1$:

- Get the pay-off of an ordinary strike-$K_1$ call-option, i.e. $(S(T_1) - K_1)^+$. 
- Extend the option. This means that by paying a predetermined amount, $A$, to the seller at time $T_1$ the buyer receives a new (standard) call
option with maturity date $T_2$ and strike $K_2$. So $A$, $T_2$, and $K_2$ are
known/specified at initiation of the contract, say time 0, and can “vary
freely”, thus affecting the time-0 price of the extendible call.

Your job now is to come up with a formula that makes it possible to actually
compute the (arbitrage-free) price of a specific extendible call. The computa-
tion could be done by simulation (OK), numerical solution of a PDE (good),
numerical integration (good), or from a a closed-form solution (excellent1).
(Hint: Look closely at what happens at $T_1$ and recall what you know about
prices of ordinary calls.)

You are not supposed to do any actual numerical calculations, but try to
device a price formula such that someone who knows programming & numerics,
but nothing about finance, can find the price. And while we’re at it you
may want to think about the following questions: What happens if $A = 0$
What happens if $K_1 \to \infty$? (This results in “an option on an option” aka. a
compound option.) Is the option always extended? Never extended? What
about hedging the option?

Exercise 9.4: Time-dependent Black/Scholes
The base-case Black/Scholes model assumes the interest rate and the volatility
are constant. You don’t have to look very hard at data to see that that is not
a particularly convincing assumption. In this exercise we look the generaliza-
tion to deterministic but time-varying parameters. This can be thought of as a
first-(or zero-)order adjustment. Perhaps not what we would ultimately want,
but it has several “spin-off stories”. More at the lectures ...

Suppose first the base-case Black/Scholes assumption of constant volatility is
maintained, but the interest rate is allowed to vary deterministically, ie. the
bank account behaves as

$$d\beta(t) = r(t)\beta(t)dt,$$

where $r$ is some smooth function. Note (prove if you have to) that this ordinary
differential equation (with 1 as initial condition) has the solution

$$\beta(t) = \exp\left(\int_0^t r(u)du\right).$$

Note also (same again . . .) that the price of a zero coupon bond, $P(t; T)$ (=
the price at time $t$ of an asset that pays 1 at time $T$), must satisfy

$$P(t; T) = \frac{\beta(t)}{\beta(T)}.$$

1Really, I’m ‘just kidding’. Don’t waste time trying to find a closed-form solution. It
is done in Longstaff (1990), “Pricing Options with Extendible Maturities: Analysis and
Applications”, Journal of Finance, vol. 45, pp 935-957. The price is a fairly complicated
expression involving the bivariate normal distribution function.
Derive a closed-form solution for the price of a call-option. Express the price in terms of (among other things, of course) zero coupon bond prices, i.e. no explicit $r$’s appearing.

Hint: It is convenient to work with the process $\tilde{S}(t) = S(t)/P(t;T)$ (find the $Q$-dynamics of $\tilde{S}$) and then to use (show!) that

$$\text{Call}(t) = P(t;T)E_Q^t((\tilde{S}(T) - K)^+).$$

Now suppose volatility is also made time-dependent (but deterministic), i.e.

$$dS(t) = r(t)S(t)dt + \sigma(t)S(t)dW^Q(t),$$

for some function $\sigma$. Derive a closed-form solution for the price of a call-option. Hint: It is still practical to work with $\tilde{S}$. Put $Y = \ln \tilde{S}$ and show that

$$dY(t) = \frac{1}{2}\sigma^2(t)dt + \sigma(t)dW^Q(t)$$

Use this & Björk’s Lemma 3.15 to conclude that $\tilde{S}(T)$ is still lognormal and find the appropriate parameters. The resulting formula should have the ordinary Black/Scholes $\sigma^2$’s replaced by

$$\frac{1}{T-t} \int_t^T \sigma^2(u)du.$$

Kindly,

Rolf