Past lectures
Monday October 27: The rest of Björk’s Chapter 9 + Section 11.1
Multidimensional models. Similarity reductions. I didn’t do the details. They
are quite tedious; later we’ll see a much easier way. But we used the result in
Proposition 9.4 to price the exchange-option (also called a Margrabe-option,
after William M. who first priced it).
Then we went on to Chapter 11 (10 comes later) where we looked at dividends.
Stocks pay out dividends; they have to, otherwise nobody would buy them
(or at least: if we are absolutely sure no money is ever received, we wouldn’t).
When pricing options, we will often “WLOG of generality assume no dividends
are paid during the life of the option”. This is “mathematically correct”, but
from a financial point of view, dividends are important.
First was the case of discrete (also called “lumpy”) dividends. Two things
happen when the dividend $\delta(t) = \delta(S(t-))$ is paid:

1. The stock price jumps down by $\delta(t)$, which is known a little bit in ad-
vance.

2. The option price doesn’t jump. No dividend is received by the option
holder, and the stock price change comes as no surprise.

This means that the option can be priced by solving a recursive system of PDEs
that are connected by so-called jump conditions at dividend dates. In the case
where the dividend is proportional to the stock price, options can be priced by
substituting $S(t)$ in the no-dividend formulas by $S(t)(1 - \delta)^{n(t)}$, where $n(t)$ is
the number of dividend dates in $[t; T]$. In the case where there are known non-
stochastic dividends, say $D(T_i)$ and we assume Black/Scholes between dates,
there is no closed form solution. This is because (stock - discounted future
dividends) does not follow a Geometric Brownian motion. But it is common
(and fairly precise) simply to use the adjusted stock price

$$S(t) - \sum_{t<T_i<T} e^{-r(T_i-t)} D(T_i)$$

in no-dividend versions of formulas.

Wednesday October 29: Rest of Chapter 11. A look at some real
data.
Continuously paid dividends, dividend yields in particular. Clearly this is a mathematical abstraction. But it’s a useful one. Good for modelling stock indices and (as we’ll see in Chapter 12) invaluable in cross-currency models. The risk-neutral (ie. \( Q \)) expected rate of return on the stock is then \( r - \delta \), which is exactly the expected rate of return on the stock a risk-neutral investor would demand because he also receives dividends. (Again: This is “as if risk-neutrality” induced by absence of arbitrage, nobody is actually assumed to be risk-neutral.)

We could establish various formulas quite easily via the artificial process

\[
\tilde{S}(t) = e^{-\delta(T-t)} S(t).
\]

This is because 1) \( \tilde{S}(T) = S(T) \) and 2) \( \tilde{S}/\beta \) is a \( Q \)-martingale. (\( \tilde{S} \) is not completely artificial; its economic interpretation is as the value of a stock portfolio where dividends are re-invested in the stock.)

The second lecture was devoted entirely to looking at real-life option data. There are lots of things that have to be considered and choices that have to be made. Often there is no perfect or unique right choice, but plenty of wrong ones! You can find a spreadsheet and some links on the homepage. You are highly encouraged to play around with these.

**Coming lectures**
Monday November 3, Wednesday November 5: I’ll do a “So Far Summary”. Slides on the homepage. And then on to currency-modelling in Chapter 12.

**Exercises for Week 45 (Wednesday November 5)**

**Exercise 8.1: Problem 1 from the 2002 exam**
Regarding the very last question: It suffices to give answers in form of “the roots of the following quadratic equation”, preferably with some indication that these are both real. The precise solutions are

\[
\beta = \gamma \pm \epsilon,
\]

where

\[
\gamma = \frac{1}{2} - \frac{r - \delta}{\sigma^2} \quad \text{and} \quad \epsilon = \sqrt{\gamma^2 + \frac{2r}{\sigma^2}}.
\]

**Exercise 8.2: Problem 2 from the 2002 exam**
Exercise 8.3: Down & Dirty with Data
Suppose we look at the Black/Scholes model for a stock that pays a constant dividend yield $\delta$. Determine

1. The put/call-parity

2. The arbitrage-free price of a call-option

Download the spreadsheet from the October 29 lectures (option analysis). Download prices of S&P500-options expiring (for instance) December 2004. How would you estimate $r$ and $\sigma$ (in the context of the model above)? How do the estimates compare to those obtained from very short-dated options (as at the lectures)? How would you estimate $\sigma$ from observed prices of the S&P500-index? How does the $\sigma$-estimate (often called historical volatility) depend on how far (1 month, 6 months, 1 year, ...) you look back? What are the implied volatilities on the options and how do they compare to historical volatility? How do implied volatilities vary across strike prices (called the ‘smile’ or ‘skew’) and how does this compare to the short-dated option skew? Hint: The spreadsheet already has used-defined functions that calculate the Black/Scholes price and implied volatility. (I pretty much stole these from Peter Raahauge.) When downloading you should click ‘enable macros’ when asked. To use the functions in formulas click on the $\text{f}$-icon and choose ‘user-defined’. They should be pretty self-explanatory, but otherwise you can full definitions (and change these) by pressing ALT-F11.

Exercise 8.4: Only for the really energetic student
Prove Björk’s Proposition 11.6.
To do this recall the risk-neutral valuation in Proposition 11.5 and define the process

$$\tilde{S}(t) = S(t)(1 - \delta)^{n(t)}$$

where $n(t)$ is the number of dividend dates in $[t; T]$. Argue that $\tilde{S}$ is continuous (what happens at dividend dates?) Argue that $\tilde{S}/\beta$ is a $Q$-martingale. Use that $\tilde{S}(T) = S(T)$ to arrive at the final conclusion. Explain why a similar argument doesn’t work in the case of fixed, deterministic (ie. non-proportional) dividends.

Kindly,

Rolf