Past lectures
Monday October 20: Björk’s Chapter 7
In the first lecture we derived the B/S-PDE once again, by turning the argument “90 degrees”. We closed a logical flaw fly in the ointment in the Chapter 6-argument. ∆-hedging was an important concept.
I then went on (and on and on) about the more general statements. Ideally we’re after the two fundamental theorems of asset pricing:

1. No arbitrage ⇔ ∃ equivalent martingale measure Q

2. An arbitrage-free model is complete if and only Q is unique.

These results are essentially also true in continuous-time models, although we can and will not prove full versions in this course. We will use the results as we see fit for the rest of the course. Essentially means that some regularity conditions have to be added. Let me say it like this: I have never seen anybody with “honest intentions” get into trouble by using the fundamental theorems of asset pricing as stated above. Problems come from not using the theorems, or using them blatantly incorrectly. If push comes to shove, you can come a long way be calculating first, and then verifying (whataver is) appropriate conditions afterwards.

An important fact to remember is that when we consider nice enough portfolios (meaning for instance that something somewhere is in Björk’s L²) then the discounted value process, V^h/β for a self-financing portfolio h is a Q-martingale. To see this, simply look at

\[
d\left(\frac{V^h(t)}{\beta(t)}\right) = \frac{dV^h(t)}{\beta(t)} + V^h(t)d\left(\frac{1}{\beta(t)}\right) + 0 \quad \text{(Ito product rule)}
\]

\[
= \frac{h^\top(t)dS(t)}{\beta(t)} - r \frac{h^\top(t)S(t)}{\beta(t)} \quad \text{(self-financing)}
\]

\[
= \left(\frac{h^\top(t)(rS(t)dt + \Sigma(t)dW^Q(t))}{\beta(t)}\right) - r \frac{h^\top(t)S(t)}{\beta(t)} \quad \text{(S/β is a Q-mg.)}
\]

\[
= \left(\frac{h^\top(t)\Sigma(t)}{\beta(t)}\right) dW^Q(t).
\]
Using this, we get the “$\leq$”-implication in the first fundamental theorem. Recall that an arbitrage is a self-financing portfolio $h$ such that:

$$V^h(0) = 0, \quad P(V^h(T) \geq 0) = 1, \quad \text{and} \quad P(V^h(T) > 0) > 0.$$ 

$\implies$

$$V^h(0) = 0, \quad Q(V^h(T) \geq 0) = 1, \quad \text{and} \quad Q(V^h(T) > 0) > 0.$$ 

$\implies$

$$V^h(0)/\beta(0) = 0, \quad Q(V^h(T)/\beta(T) \geq 0) = 1, \quad \text{and} \quad Q(V^h(T)/\beta(T) > 0) > 0.$$ 

But we just saw that $V^h/\beta$ is a $Q$-martingale, so $E^Q(V^h(T)/\beta(T)) = V^h(0) = 0$, which clearly contradicts the last equation above. The converse implication goes something like this: Absence of arbitrage implies the existence of a strictly positive linear functional on the (linear) space of cash-flows generated by trading strategies. This can be Riesz represented.

So if we have a complete and arbitrage-free model where $Q$ is an equivalent martingale measure, then for any traded asset, the unique arbitrage-free price is given by

$$\pi(t) = \beta(t)E^Q_t(\pi(T)/\beta(T)),$$

also (with the same $Q$) for whatever new contingent claim we may introduce. Note the logic: It says “There $\exists Q$ such that $\forall \pi \ldots$” so it’s the same measure for all $\pi$’s.

Completeness is arguably the most interesting concept; it is what allows us to give definite statements about the prices of options we might want to introduce into the model. Mathematically, the question of completeness and replication/hedging is linked to so-called martingale representation results. The martingale representation result says that any $\mathcal{F}_T^W$-measurable square integrable random variable $X$ be written as a stochastic integral wrt. Brownian motion, ie. there exists a (unique) process $\phi$ such that

$$X = E(X) + \int_0^T \phi(u)dW(u).$$

(We couldn’t hope for $E(X)$ not to appear on the right-hand-side of the equation (if $\phi$ is to be nice). Explain to yourself why, if need be.) The result is “non-constructive”; it doesn’t tell us what $\phi$ looks like. But for simple variables, $X = G(W(T))$, we can find it via an argument similar to the one used when $\Delta$-hedging. At the risk sounding smug: If you have the right tools, martingale representation results are easier to prove than you should think, see for instance Chapter 4 in the aforementioned Øksendal-book.

It should be clear that martingale representation results are closely related to replication via self-financing strategies; the question is whether or not prices generate the same information as the Brownian motion.
So how do we determine if a given model is complete? Well, we can trust the second fundamental theorem of asset pricing, and look for different equivalent martingale measures by means of Girsanov’s theorem. Or we can trust Bjöörk’s useful rule of thumb: “For completeness, the number of stocks must equal the number of sources of risk involved (typically the dimension of the Brownian motion).”

**Wednesday October 8: Bjöörk’s Chapter 8 and Sections 9.1-3**

Chapter 8 is not theoretically deep, but it contains many things that are good to think about and experiment with. I showed some numerical results for the discrete Δ-hedging experiment Björk describes on page 114. Code, graphs and some related questions for the curious student can be found on the homepage. If you want to learn more or write a project then come and ask me; there are tons of things to investigate.

Chapter 9 is about multi-dimensional models. Björk spends about 10 pages deriving a PDE & then risk-neutral valuation. This is absolutely correct but with the fundamental theorems and Girsanov in place we can do things much easier. The argument goes like this: In matrix/vector-notation consider the model

\[
\frac{dS_i}{S_i} = \mu dt + \sigma dW^P_i,
\]

where \( I_S = \text{diag}(S_1, \ldots, S_n) \). If \( S_i/\beta \) is to be a martingale, then its drift must be \( rS_i \). An equivalent martingale measure is then one that changes the drift from \( I_S \mu \) to \( I_S r \). Now we apply Girsanov’s theorem with \( \alpha = I_S r \) and \( \theta = I_S \sigma \). It tells us that to determine a \( Q \) we must find a \( u \) (a \( d \)-dimensional stochastic process) that (for each \( t \) and \( \omega \) it things are time-dependent and stochastic) solves

\[
I_S \sigma u = I_S \mu - I_S r \Rightarrow \sigma u = \mu - r1.
\]

If \( n = d \) (do the arguments for the other cases yourself) and \( \sigma \) has full rank (and is therefore invertible) then there exists precisely one such \( u \) (\( u = \sigma^{-1}(\mu - r1) \)) and hence a unique equivalent martingale measure. So the model is arbitrage-free and complete, and we can price any new contingent claim in a unique way because we can replicate it.

Now, in the diffusion case and with a simple claim, we can easily write down a (high-dimensional) PDE by combining the martingale property with the multi-dimensional Itô formula.

**Coming lectures**

Monday October 27, Wednesday October 29: We’ll finish Chapter 9 and then go to Chapter 11 (we’ll get back to 10 later), where we look at dividends.
Exercises for Week 44 (Wednesday October 29)

Exercise 7.1:
Prove that $\Delta$ for a call-option in the Black/Scholes model is indeed given by

$$\Delta(t) = \Phi(d_1) = \Phi\left(\frac{\ln(S(t)/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}\right),$$

as stipulated in Björk’s Proposition 8.5 (and lots of other places).
Hint: Careful! $S$ appears inside $\Phi$, so you need the chain rule when differentiating.

Exercises 7.2-4:
Questions 2a, 2b and 2c from Exercise 2 in the Winter 2002-exam. I’ve posted the exam set on the homepage.

Exercise 7.5:
Björk’s Exercise 9.3. (I’ll do the $\rho = 0$ case at the lectures on Monday.)

Kindly,

Rolf