Mathematical Finance, Fall 2003, Weeks 41 & 42

Past lectures
Monday October 6: Björk Section 6.4 and the Øksendal hand-out
Representation of arbitrage-free prices as expected discounted pay-offs under a new probability measure $Q$. Under this measure discounted prices are martingales, and therefore $Q$ is called an equivalent martingale measure.
The main result from Øksendal: Girsanov’s Theorem. In words, it says that by changing between equivalent probability measures, we can change the drift of an Ito process into anything we like, but we cannot change to volatility. Notice that the new measure and the new Brownian motion are quite explicitly characterized (but that often, we don’t really need that).
Wednesday October 8: The rest of Björk’s Chapter 6
Twists of the Black/Scholes formula/model: Options on forwards (& futures). Volatilities (historically estimated and implied). American options (that make a nice topic for projects or even theses; I’ll put a more detailed on the home-page).

Coming lectures
Week 42 is fall break (“efterskole”), so there are no lectures or exercises. On Monday October 20, I’ll start on Chapter 7 (that will be covered fairly quickly).

Exercises for Week 43 (Wednesday October 22)

Exercise 6.1:
Solve Björk’s exercise 6.1.

Exercise 6.2:
Solve Björk’s exercise 7.2.
Exercise 6.3:
Solve Björk’s exercise 7.3.
By using the martingale formulation of the price,
\[ \pi(t) = e^{-r(T_2-t)} \mathbb{E}_t^Q \left( \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(u) du \right), \]
and interchanging expectation and integration (possibly after splitting up the integral) and using the properties of \( Q \), you can derive a price-formula without using the explicit form of Geometric Brownian motion. In fact, you arrive at a result that is valid irrespective of what the volatility is, just as long as interest rates are deterministic.

Exercise 6.4: New measures
Consider a finite probability space \((\Omega, \mathcal{F}, P)\), and a random variable \( X \) such that \( X(\omega) > 0 \) for all \( \omega \) and \( \mathbb{E}(X) = 1 \). Define \( Q \) by putting \( Q(A) = \mathbb{E}(1_A X) \) for any event \( A \), ie. \( A \in \mathcal{F} \) (with \( 1 \) denoting the indicator function).

Remind yourself why
- \( Q \) is a probability measure
- \( Q(A) = 0 \iff P(A) = 0 \), ie. \( Q \) and \( P \) are equivalent
- for any random variable \( Z \) we have
\[ \mathbb{E}^Q(Z) = \mathbb{E}^P(ZX), \]
and that explains why we use the seemingly odd “fractional notation” \( \frac{dQ}{dP} = X \) (this quantity is called the Radon/Nikodym derivative of \( Q \) wrt. \( P \)).

Show that \( \mathbb{E}^Q(\frac{1}{X}) = 1 \) and note that that can be used to introduce yet another probability measure, say \( \bar{Q} \), in similar fashion as above. Show that in fact \( \bar{Q} = P \), and
\[ \mathbb{E}^P(Z) = \mathbb{E}^Q \left( \frac{Z}{X} \right), \]
which further justifies the fractional notation because then \( \frac{dP}{dQ} = \frac{1}{X} = \left( \frac{dP}{dQ} \right)^{-1} \).
Exercise 6.5:
Show why we have the arbitrage-bound

$$\pi^{EU-CALL}(t) \geq S(t) - e^{-r(T-t)}K$$

for a European call-option on a non-dividend paying stock. Use this to argue that when $r > 0$, American calls should never be exercised before expiry. (Note that the argument works only for calls & in the no-dividend case. When there are dividends, early exercise may be optimal. And early exercise for puts may be optimal.)

Extra exercises
On the homepage, I have posted to extra exercises. These are completely voluntary, and you should not spend time on them in class. But you should try to solve the if a) you feel you need to or b) you are curious.

EE: Conditional Expectation is about (surprise) conditional expectation. Farhang says that if you hand in answers to him, then he’ll correct them.

EE: Importance Sampling shows that sometimes Girsanov’s theorem can be used to greatly improve the efficiency of simulation. This is not how we usually use Girsanov, and using such a general result about Ito process to deal with a single normal variable is a tremendous “overkill” (like when we used Ito’s formula to calculate the 4th moment of a normal variable). I just think it’s cute.

Kindly,

Rolf