Mathematical Finance, Fall 2003, Week 36

Practical matters
As you probably already know: Lectures take place Mondays and Wednesdays 8-10 in Aud. 10. As text-book we will use the nice book ‘Arbitrage Theory in Continuous Time’ by Tomas Björk, which we will go through pretty much cover to cover.
Besides lectures, our main means of communication will be the course home-page
http://www.math.ku.dk/~rolf/teaching/mfe03/mfe03.html

There are weekly 2 hours of exercise classes (’)ovelser’) with Farhang Farazmand as teaching assistant (’instruktor’).
So far, the classes are scheduled to take place on Wednesdays 10-12 in room A104. This time is not set in stone; I understand that it is quite inconvenient for several people. In week 37 the exercises follow the original plan, but one of your tasks (and I do mean yours) is to find the optimal time for the classes.

The exam will be a 1-day ‘take-home-exam’ (hopefully this lowers the stress factor). You sign up for the exam in the (if you’re mat-ok’er) usual electronic way. Other people sign up at a form they get from Studie- og Eksamenskontoret at the Faculty of Science. The sign-up period is late September, early October. But don’t worry, I’ll keep you informed. The exact exam-dates have not been set yet.

Past lectures
Monday September 1
Introduction, practical stuff, and a reminder of binomial models. I covered Björk chapters 1 & 2 (although in double-quick-time at the end).

Wednesday September 3
I covered the first few pages of Chapter 3. We met Brownian motion aka (= ’also known as’) the Wiener process. It is a fundamental building block for continuous time stochastic processes. I mentioned some of its more or less weird properties; a non-weird one is that $\text{cov}(W_s, W_t) = \min(s, t)$. There was some mention of the concept of “information”. 
Coming lectures
Monday September 8 and the following lectures: More along the beaten path in Chapter 3.

Exercises for Week 37 (Wednesday September 10)

Figure out when the exercises take place in the future.

Björk’s Chapter 2-exercises are somewhat tedious, but it’s not a bad idea to look at them if you think I went over Chapter 2 a bit too fast. However, in class you should concentrate on these (where 1.2 & 1.3 are the important ones):

Exercise 1.1:
Comment on Definition 2.2 (of an arbitrage) in Björk. Is this the definition we used in Investments and Finance Theory? (Would free lottery tickets be arbitrage by this definition?) And how does the definition affect the statement in Proposition 2.3?

Exercise 1.2: Simulating (not) Brownian motion
The right way: Use your favorite software to simulate Brownian motion (at time points $t_i$) using this algorithm:

- Draw $X_i$’s that are independent Normal(0,1)
- Put $W(t_i) = W(t_{i-1}) + \sqrt{t_i - t_{i-1}}X_i$

Plot some sample paths. “Zoom in” on a particular path; what does it look like? (Strictly speaking, you cannot plot the full sample path, but only some points on it. Joining the points with straight lines is “cheating”. Do it anyway.)

The wrong way: Consider the following pseudo-code for simulating Brownian motion:

- Draw $X_i$’s that are independent Normal(0,1)
- Put $W(t_i) = \sqrt{t_i}X_i$

Try it & plot some sample paths. Compare to “the right way”; be sure to use small time steps. Explain which part of the definition of Brownian motion the process violates.

The wrong way, but faster (Simpsons aficionados will recognize this as “the Max Power way”, see http://www.snpp.com/episodes/AABF09): Consider the following pseudo-code for simulating Brownian motion:

- Draw an $X$ that is Normal(0,1)
- Put $W(t_i) = \sqrt{t_i}X$

Try it & plot some sample paths. Which part of the BM-definition is violated this time?
(If you have never heard of the Poisson-process, then drop this question. Feel free to drop it anyway.)

Not Brownian motion either: Let $N = (N_t)_{t \geq 0}$ be a Poisson-process with intensity $\lambda$, and define the process $M$ by $M_t = (N_t - \lambda t)/\sqrt{\lambda}$. Find the mean and covariance functions for $M$. Does $M$ have independent increments? Is $M$ a SBM? How would you simulate sample paths of $M$? (Hint: If $U \sim U(0,1)$ then $-\ln U/\lambda \sim \text{Exp}(\lambda)$.) What do the paths look like?

**Exercise 1.3: Other Brownian motions**

In the following $W = (W_t)_{t \geq 0}$ denotes a standard Brownian motion (SBM), typically 1-dimensional. When we say that something is a SBM we mean wrt. its own filtration.

Let $s \in \mathbb{R}_+$ and show that the process $(W_{s+t} - W_s)_{t \geq 0}$ is a SBM.

Let $s \in \mathbb{R}_+$ and show that the process $(W_{s-t} - W_s)_{0 \leq t \leq s}$ is a SBM restricted to $[0; s]$ (which means what you think it means).

Let $a \in \mathbb{R}_+$ and show that the process $(\sqrt{a}W_{t/a})_{t \geq 0}$ is a SBM.

Show that the process defined by

$$X_t = \begin{cases} tW_{t/t} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

is a SBM. (It’s a bit tricky showing continuity at 0. If you can use the law of large numbers to argue that for $n \in \mathbb{N}$ we have $\lim_{n \to \infty} X_{1/n} = 0$ almost surely, it’ll suffice for this course. The continuity of the time parameter leaves a little gap in the argument. If you want a full proof you can use that $Z(n) = \max_{0 \leq s \leq 1} |W(s + n) - W(n)|$ defines independent, identically distributed random variables that (by something called Doob’s inequality) have finite mean. Because $Z(n) = \sum_{i=1}^{n} Z_i - \sum_{i=1}^{n-1} Z_i$, $Z(n)/n$ converges a.s. to 0, and the result follows by a little mingling around. **Do not waste time with this last part in class.**)

Let $W = (W_t^{(1)}, W_t^{(2)}, \ldots, W_t^{(d)})^\top$ be a (column) vector of independent SBMs and suppose $\alpha \in \mathbb{R}^d \setminus \{0\}$. Show that $\alpha^\top W/\|\alpha\|$ defines a SBM. ($\| \cdot \|$ is the standard Euclidic norm, ie. $\|\alpha\| = \sqrt{\sum \alpha_i^2}$.)

**Exercise 1.4: Does more data help?**

Suppose the process $X$ is defined by

$$X_t = \mu t + \sigma W_t,$$
where $W$ is a SBM. Such a process is called Brownian motion with drift (but, for some reason, never Wiener process with drift).

The question is now: What happens if we do not know the values of $\mu$ and $\sigma$, but have to estimate them? To this end, suppose that an interval $[0; T]$ has been split into $n$ pieces of equal size ($=\Delta(n) = T/n$) at points $t_i$, $i = 0, \ldots, n$, and that we have observed a realization of $X$ at these points, i.e. we have observations $x_i^{(n)}$ of $X_t$. Put $r_i^{(n)} = x_i^{(n)} - x_{i-1}^{(n)}$.

Which distribution do the $r_i^{(n)}$’s come from? Argue that sensible estimators for the mean and variance of the $r_i^{(n)}$’s are

$$\bar{r}^{(n)} = \frac{1}{n} \sum_{i=1}^{n} r_i^{(n)} \quad \text{and} \quad (s^{(n)})^2 = \frac{1}{n-1} \sum_{i=1}^{n} (r_i^{(n)} - \bar{r}^{(n)})^2 \quad \text{(or use } 1/n \text{ instead; I don’t care)}.$$

So how would you estimate $\mu$ and $\sigma^2$? What are the standard errors of the estimators? (For $\hat{\mu}$ you should know it or find it easily. Regarding $\hat{\sigma}^2$ you may not know it explicitly. I can then tell you that $\text{std.dev.}((s^{(n)})^2) = \sqrt{\text{var}(r_i^{(n)})}/\sqrt{n-1}$.) What do confidence intervals look like?

Now suppose we sample data more and more frequently, i.e. we let $n \to \infty$, but we keep $T$ fixed. What happens to the mean and variance of the $r_i^{(n)}$’s? What happens to standard errors of the estimators of $\mu$ and $\sigma^2$? What does that say about the answer to the question in the title?

Kindly,

Rolf