where \( P(t, T) = \exp \left( -\int_t^T r(u) du \right) \) is the zero coupon bond price and
\[
\sigma^2_{AV} = \frac{1}{T-t} \int_t^T \sigma^2(u) du.
\]

This is (sometimes) called the Black/Scholes/Merton formula.

Not too hard to prove with martingale methods; hard to guess with PDEs.

Deterministic changes may not be what we ultimately want, but it’s a first step. “Shades of the real world” & gives useful information.

### Volatility
What matters is the average (or integrated) (squared) volatility during the life of the option. “Future volatility, not past volatility”.

If volatility is stochastic, \( \sigma(t, \omega) \), then the random variable \( \int_t^T \sigma^2(u, \omega) du \) still plays an important role for option pricing, although things are quite as nice as in the \( r \)-case.

If \( \sigma \) is exogenous to the stock price (driven by another BM) then by conditioning:
\[
\text{call}(t) = \int \text{call}^{\text{BSM}}(\ldots, \sigma^2_{AV} = x, \ldots) \phi_{\sigma^2_{AV}}(x) dx,
\]
where \( \phi_{\sigma^2_{AV}} \) denotes the density of \( \frac{1}{T-t} \int_t^T \sigma^2(u, \omega) du \). Expansions lead to (approximate) formulas. (Hull & White 1987.)

When \( r \) and \( \sigma \) are constant \( \sim \) original B/S formula. Time-varying dividend yields: DIY (relevant in currency models).

### Interest rates
If there is a term structure of interest rates, then in place of \( r \) in original B/S formula you should plug in the zero coupon rate with maturity equal to the time to expiry of the option:
\[
r \sim y(t, T) \quad (\text{since by definition} P(t, T) = \exp(-y(t, T)(T - t)))
\]

With stochastic interest rates, the ZC yield \( y(t, T) \) can also be plugged in. A good approximation for stocks. Can be theoretically supported when interest rates are Gaussian & independent of the stock.
The LHS can be estimated from data by summing squared log-returns over the observation period $[T_i; T_i]$. This is called \textit{realized} volatility.

If the volatility path is known to market participants, then the BSM-analysis tells us that the (Black/Scholes) \textit{implied volatility} of an option (after scaling) is $\int_{T_i}^{T_{i+1}} \sigma^2(u, \omega)du$. This can be recorded/observed already at time $T_i$.

This leads to the empirical/econometric question: Is implied volatility a good forecast for realized volatility? In particular, does implied volatility contain information that can't be extracted from past/historical data?

\textbf{Other Spin-Off: Implied vs. Realized Volatility}

If $\sigma$ is stochastic and

$$dS(t) = S(t)\mu(t)dt + \sigma(t, \omega)S(t)dW^P,$$

then with $Y = \ln S$ we have

$$dY = (\mu - \sigma^2/2)dt + \sigma(t, \omega)dW^P$$

Now look at some $[T_i; T_{i+1}]$ and suppose it has been split into $n$ pieces (intermediate points $t_j$).

Then from our analysis of quadratic variation we know that

$$\sum_j (Y(t_{j+1}) - Y(t_j))^2 \overset{n \to \infty}{\to} \int_{T_i}^{T_{i+1}} \sigma^2(u, \omega)du$$

(All objects are random variables; convergence is in $L^2$-sense.)

Empirical analysis: Record data $\to$ time series of realized volatilities, $\sigma_{AV}(t_k)$ and corresponding implied volatilities $\sigma_{IM}(t_k)$.

Run regression (possibly in logs)

$$\sigma_{AV}(t_k) = \alpha_0 + \alpha_1 \sigma_{IM}(t_k) + \epsilon(t_{k+1})$$

If $\alpha_1 \neq 0$, then implied volatility contains some information.

If $(\alpha_0, \alpha_1) = (0, 1)$ then implied volatility is an unbiased forecast.

Include eg. $\alpha_2 \sigma_{AV}(t_{k-1})$; test if $\alpha_2 = 0$.

Results: There is some information content in implied volatility that historical volatility does not capture. Beware of overlapping data!