EXERCISE 1; STOCHASTIC CALCULUS

In this exercise "everything is 1-dimensional" and $W$ denotes a (standard) Brownian motion.

1a [10%]
Let $x_0$ be a real number, and $a$, $b$ and $c$ be sufficiently smooth deterministic functions. Consider the process $X$ defined by

$$X(t) = a(t) \left( x_0 + \int_0^t b(s) \, ds + \int_0^t c(s) \, dW(s) \right).$$

Explain why $X(t)$ is normally distributed for all $t$. (Not surprisingly, it can be shown that $X$ is in fact a Gaussian process.)

Show that if $a(t) > 0$ for $t \geq 0$, then the dynamics of $X$ can be written as

$$dX(t) = \left( a(t)b(t) + \frac{a'(t)}{a(t)} X(t) \right) \, dt + a(t)c(t) \, dW(t).$$

Hint: Write $X$ as $X(t) = a(t)B(t)$ for a suitably defined $B$-process.
1b[10%]
Now consider the SDE
\[ dX(t) = -\frac{X(t)}{1-t}dt + dW(t), \quad X(0) = 0 \] (1)
on the interval \([0; 1]\). Clearly “\(t = 1\)” in the equation above is problematic, so let’s stay away from there for the time being.
Use Question 1a to conclude that the solution to (1) is
\[ X(t) = (1 - t) \int_0^t \frac{1}{1-u} dW(u) \quad \text{for } t \in [0; 1[. \]
Show that \( E(X(t)) = 0 \). Is \( X \) a martingale?
Show that \( \text{cov}(X(s), X(t)) = s(1-t) \) for \( 0 \leq s \leq t < 1 \) and explain why defining \( X(1) = 0 \) is very reasonable.
Hint: To determine the covariance, look at
\[
E \left( \int_0^s \frac{dW(u)}{1-u} \times \int_0^t \frac{dW(u)}{1-u} \right) = E \left( \int_0^s \frac{dW(u)}{1-u} \times \left( \int_0^s \frac{dW(u)}{1-u} + \int_s^t \frac{dW(u)}{1-u} \right) \right),
\]
and use fundamental properties (“martingality”, Ito isometry) of the stochastic integral.

A process with distributional characteristics as \( X \) is called a Brownian bridge.

1c[10%]
Answer one (and only one) of the following questions (labeled i) and ii)).
Hint: If you’re unfamiliar, or uncomfortable, with conditioning for normal variables — as seen by mat/\$k\$’er in Statistics 1MS/TS/A — then question i) is recommended.

i) Show that the process defined by
\[ Y(t) = W(t) - tW(1) \quad \text{for } 0 \leq t \leq 1 \]
is a Brownian bridge, i.e. show that it is a 0-mean Gaussian process with covariance function \( \text{cov}(Y(s), Y(t)) = s(1-t) \) for \( 0 \leq s \leq t \leq 1 \).
This is useful for simulation of Brownian bridges.

ii) Show that the process defined by
\[ Z(t) = W(t)\mid W(1) = 0 \quad \text{for } 0 \leq t \leq 1 \]
is a Brownian bridge, ie. show that it is a 0-mean Gaussian process with covariance function \( \text{cov}(Z(s), Z(t)) = s(1 - t) \) for \( 0 \leq s \leq t \leq 1 \).

This shows that a Brownian bridge can be thought of as regular Brownian motion “pinned down” at 0 at time t. In a similar way other Bridge-type processes can be defined.

**EXERCISE 2; BLACK/SCHOLES OPTION PRICING**

In this exercise we consider the base-case Black/Scholes model.

2a [10%]
Suppose we introduce a new contract: The Square. This contract has the pay-off

\[ S^2(T) \text{ at time } T. \]

Show that the time-\( t \) arbitrage-free price of the Square is

\[ \pi^{\text{SQ}}(t) = S^2(t)e^{(r+\sigma^2)(T-t)}. \]

Explain how to hedge/replicate the contract.

2b [10%]
Suppose we introduce a new contract: The chooser option. This option allows the holder to chose at time \( T_1 \) to receive either an expiry-\( T_2 \) strike-\( K \) call-option, or a similar put-option (ie. expiry-\( T_2 \) strike-\( K \)).

Show that the time-\( t \) (\( t \leq T_1 \)) arbitrage-free price of the chooser option is

\[ \pi^{\text{chooser}}(t) = \pi^{\text{call}}(t|\text{strike } = K, \text{expiry } = T_2) + \pi^{\text{put}}(t|\text{strike } = Ke^{-r(T_2-T_1)}, \text{expiry } = T_1), \]

where \( \pi^{\text{call/put}} \) denote prices of an ordinary call/put.

Hint: Use the put/call-parity (at time \( T_1 \)).

If the put and call that the holder can choose between do not have the same strikes and expiry dates, then things do not go this easily. Explain briefly how you would then calculate the price of the option. (You are not supposed to find a closed-form solution, nor do any actual numerical calculations. Just sketch a method that would work. There are several ways to do this.)
EXERCISE 3: OPTION PRICING WITH TWISTS

In this exercise, the interest rate is 0. (Alternatively, you can think of all statements being made in terms of discounted stock-prices.)

3a [10%]
Suppose now that we model the stock-price, \( S \), not by Geometric Brownian motion as in the Black/Scholes model, but instead by ordinary Brownian motion, i.e. since interest rates are 0 we have that

\[
S(t) = S_0 + \sigma W^Q(t).
\]

This is called the Bachelier model. This is after Louis Bachelier, a French pioneer in finance and stochastic processes who used it in his 1900 dissertation. Consider a strike-\( K \) expiry-\( T \) call-option. Show that its arbitrage-free time-\( t \) price is

\[
\pi_{\text{call, Bach}}(t) = (S(t) - K)\Phi \left( \frac{S(t) - K}{\sigma \sqrt{T - t}} \right) + \sigma \sqrt{T - t} \phi \left( \frac{S(t) - K}{\sigma \sqrt{T - t}} \right),
\]

where \( \Phi \) and \( \phi \) denote, respectively, the standard normal distribution and density function, i.e.

\[
\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \phi(u) du.
\]

Show that the time-\( t \) \( \Delta \)-hedge-ratio of the call-option is

\[
\Delta_{\text{call, Bach}}(t) = \Phi \left( \frac{S(t) - K}{\sigma \sqrt{T - t}} \right).
\]

Hint: You may, without proof, use this result about normal distributions:
If \( X \sim N(\mu, \sigma^2) \) then

\[
E(X1_{l \leq X \leq h}) = \mu \left( \Phi \left( \frac{h - \mu}{\sigma} \right) - \Phi \left( \frac{l - \mu}{\sigma} \right) \right) + \sigma \left( \phi \left( \frac{l - \mu}{\sigma} \right) - \phi \left( \frac{h - \mu}{\sigma} \right) \right),
\]

where as usual \( 1 \) denotes the indicator function.

3b [10%]
Recall the general result about call-option pricing (simplified to “\( r = 0 \)” and “\( t = 0 \)”)
that states that

\[
\text{Call}(0) = S(0)Q^S(S(T) \geq K) - KQ(S(T) \geq K),
\]

where \( Q^S \) denotes the martingale measure with \( S \) as numeraire.
From this it is very tempting to conclude that the $\Delta$ of a call-option is

$$\Delta^{\text{call}}(0) \overset{?}{=} Q(S(T) \geq K). \quad (3)$$

The problem is, though, that the probability-terms also depend on $S(0)$ (because $S(T)$’s distribution does).

Show that in Black/Scholes model, ‘yes’, equation (3) does indeed hold.

Hint: Remember that you know what the left hand side of (3) is in this case.

3c [10%]
Answer one (and only one) of the following questions (labeled $i$) and $ii$).

$i$) What do the (Black/Scholes) implied volatilities of the Bachelier model look like? Comment on the results. ($S(0) = 100$, $T = 1$ and $\sigma = 15$ are reasonable parameters.)

$ii$) Using the Bachelier model as a counterexample, show that ‘no’, equation (3) is not generally true. Hint: Isolate $Q^S(\cdot)$ in (2) and compare to the result from the first part of the exercise. (Do not worry about the fact that dividing by Brownian motion is dubious.)

**EXERCISE 4: INTEREST RATES**

In this exercise we look at the Vasicek model, which we interpret in the following way:
Under $P$ (the real-world probability measure) the short rate dynamics are

$$dr(t) = \kappa(\theta^P - r(t))dt + \sigma dW^P(t),$$

with some initial short rate $r_0$. Under suitable assumptions, the shift to a martingale measure, $Q$, can be subsumed in a change in the $\theta$-parameter,

$$\theta^Q = \theta^P + \lambda,$$

for some real number $\lambda$. Under $Q$ the dynamics are then

$$dr(t) = \kappa(\theta^Q - r(t))dt + \sigma dW^Q(t).$$

Should you feel you “need numbers” (a legitimate way to answer some of the questions), then $\theta^P = 0.04$, $\theta^Q = 0.06$, $\kappa = 0.5$, and $\sigma = 0.01$ are reasonable values.
4a [10%] 
Show that the $Q$-expected future short rates are 

$$E^Q(r(T)) = r_0 e^{-\kappa T} + \theta^Q (1 - e^{-\kappa T}).$$

What are the forward rates, $f(0, T)$? 
Are $Q$-expected future short rates and forward rates equal?

4b [10%] 
Suppose $r_0 = \theta^Q$. Then what does the term-structure/yield curve look like? (A detailed investigation is not required; just the big picture.)
This we can interpret as the "$Q$-typical"-shape of the yield curve, or as the typical shape in a risk-neutral world.
Suppose $r_0 = \theta^P$ and that $\lambda > 0$. Now what is the shape of the yield curve? With similar reasoning, this can be called the "$P$-typical"-shape of the yield curve.
Comment on the results.