EXERCISE 1

1a [10%]
With $f(x, t) = x^3/3 - tx$, we have $f_x = x^2 - t$, $f_{xx} = 2x$, $f_t = -x$, and $Y(t) = f(t, W(t))$. Of course, $W$ has 0 drift and unit volatility, so by the Ito-formula we find that

$$dY(t) = (f_t(t, W(t)) + \frac{1}{2}f_{xx}(t, W(t))) + f_x(t, W(t))dW(t)$$

$$= (W^2(t) - t)dW(t).$$

The differential contains only $dW$-terms (i.e. no drift- or $dt$-terms), so by Björk’s Corollary 3.8, $Y$ is a martingale. (The integrand $W^2(t) - t$ fulfills the integrability conditions that “purists” may want, since clearly $\int_0^T E((W^2(u) - u)^2)du < \infty$ for all $T$.)

1b[15%]
Interchanging mean & integration (aka applying Fubini’s Theorem) we get

$$E(Z(t)) = \int_0^t E(W(u))du = 0 \text{ for all } t.$$ (We are “only” interchanging two ordinary integrals, so “purists” wanting to check regularity conditions should be able to do that. Fubini’s Theorem has quite mild regularity conditions.)

Collecting terms on the RHS of (1) we get

$$\int_0^t \int_0^s W(u)W(s)duds + \int_0^t \int_s^t W(u)W(s)duds = \int_0^t \int_0^s W(u)W(s)duds$$

$$= \int_0^t W(s) \left(\int_0^s W(u)du\right) ds$$

$$= Z(t) \int_0^t W(s)ds = Z^2(t)$$

We know from the “Week 36-37”-note that $\text{cov}(W(u), W(s)) = \min(s, u)$ (it’s easy to prove anyway). So we get

$$\text{Var}(Z(t)) = E(Z^2(t)) = \int_0^t \int_0^s E(W(u)W(s))duds + \int_0^t \int_s^t E(W(u)W(s))duds$$

$$= \int_0^t \frac{s^2}{2} ds + \int_0^t (t-s) ds$$

$$= \frac{t^3}{6} + \frac{t^2}{2} - \frac{t^3}{3} = \frac{t^3}{3} \text{ for all } t.$$
EXERCISE 2: OPTION PRICING

2a [10%]
The short & perfectly acceptable answer is to use the 1999-exam questions (with change of sign) to get $\pi(t)$. The slightly longer version goes like this:

\[
\pi(t) = e^{-r(T-t)}E_t^Q \left( S(T) - \frac{1}{T} \int_0^T S(u)du \right)
\]

\[
= S(t) - \frac{e^{-r(T-t)}}{T} \int_0^t S(u)du - \frac{e^{-r(T-t)}}{T} \int_t^T E_t^Q(S(u))du
\]

\[
= S(t) - \frac{e^{-r(T-t)}}{T} \int_0^t S(u)du - \frac{e^{-r(T-t)}}{T} \int_t^T e^{r(u-t)}E_t^Q(e^{-r(u-t)}S(u))du
\]

Putting $Z(t) = \int_0^t S(u)du$, we have that

\[
\pi(t) = f(t, S(t), Z(t)),
\]

where $f(t, x, z) = x(1 - (1 - e^{-r(T-t)})/(rT)) + e^{-r(T-t)}z/T$. From Björk’s Proposition 7.6 we get that a replicating strategy has

\[
f_x(t, S(t), Z(t)) = 1 - \frac{1 - e^{-r(T-t)}}{rT}
\]

units of stock at time $t$ and is kept self-financing by appropriate trading in the bank-account. A priori simple $\Delta$-hedging might appear not to work because the price depends also on past stock-prices. But by Proposition 7.6 it does, the reason being that the “extra stochastic factor” is of a particularly simple form.

2b [10%]
Evidently, we get as in the usual Black-Scholes calculation

\[
\pi^{c/n}(T) = e^{-r(T-t)}E_t^Q(1_{S(T)\geq K})
\]

\[
= e^{-r(T-t)}\Phi \left( \frac{\ln(S(t)/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right).
\]

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2c [20%]

The replicating strategy is given by

\[
f_x(t, S(t), Z(t)) = 1 - \frac{1 - e^{-r(T-t)}}{rT}
\]

where $x = e^{-r(T-t)}S(t)$. The self-financing condition is satisfied by the replicating strategy. The Black-Scholes formula for the option price is

\[
\pi^{c/n}(T) = e^{-r(T-t)}\Phi \left( \frac{\ln(S(t)/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right).
\]
We may rewrite
\[ d_- = \ln \left( \frac{S(t)e^{r(T-t)}}{K} \right) \frac{1}{\sigma \sqrt{T-t}} - \frac{1}{2} \sigma \sqrt{T-t}, \]
from which it is clear that
\[ \frac{\partial d_-}{\partial \sigma} = -\ln \left( \frac{S(t)e^{r(T-t)}}{K} \right) \frac{1}{\sigma^2 \sqrt{T-t}} - \frac{1}{2} \sqrt{T-t} = -\frac{d_-}{\sigma} - \sqrt{T-t}. \]

If \( S(t) \) is small, then \( \frac{\partial d_-}{\partial \sigma} \) is positive, whereas it’s negative for large stock prices. Let \( s^* \) be the “critical price”, i.e. the stock price such that \( \frac{\partial d_-}{\partial \sigma} = 0 \). We find that
\[ s^* = Ke^{-r+\sigma^2/2)(T-t)}. \]

(It’s not important to find the critical point explicitly.) By the chain rule we have
\[ \text{Vega}_{c/n} = e^{-r(T-t)} \phi(d_-) \frac{\partial d_-}{\partial \sigma} \]
so the sign of Vega_{c/n} follows the sign of \( \frac{\partial d_-}{\partial \sigma} \) (and for very large or small stock prices, the \( \phi \)-term kills it). In words, if the option is (somewhat) out-of-the-money, then “more volatility is good” because it’s the only thing that can bring the stock price to a level where the option pays off positively (and further drops in the stock price don’t matter, we can’t get less than 0). On the other hand, if the option is in-the-money, then “volatility is bad” because it makes it more likely that the option finishes worthless & we get “nothing extra” whether the option expires deep in-the-money, or just a tiny bit. For an ordinary call, Vega is always positive, intuitively because the added effect of the option paying a great deal more for very large stock prices is strong enough to make “volatility a good thing”.

\[ \text{2c: [15%]} \]
Computing the square we get
\[ \pi_{pc}(T) = e^{-r(T-t)} \mathbb{E}_t^Q \left( S^2(T)1_{S(T) \geq K} \right) - 2Ke^{-r(T-t)} \mathbb{E}_t^Q (S(T)1_{S(T) \geq K}) + K^2e^{-r(T-t)} \mathbb{E}_t^Q (1_{S(T) \geq K}). \]

We immediately recognize the two last terms as “old friends” from the Black-Scholes calculation (multiplied by \(-2K\) and \(-K\), respectively). So the two last terms of the stated formula are OK, only the first one is non-standard. But note that
\[ S^2(T) = S^2(t) \exp(2(r - \sigma^2/2)(T-t) + 2\sigma(W^Q(T) - W^Q(t))) = S^2(t) \exp(2Z), \]
where \( Z \sim N((r - \sigma^2/2)(T-t), \sigma^2(T-t)) \) and independent of \( \mathcal{F}_t \). We also have \( S(T) = S(t) \exp(Z) \) so
\[ \mathbb{E}_t^Q (S^2(T)1_{S(T) \geq K}) = S^2(t) \mathbb{E}_t^Q (e^{2Z}1_{Z \geq \ln(K/S(t))}). \]
Now apply **Homemade Exercise 4.1** with

\[
h = +\infty, \quad l = \ln(K/S(t)), \quad a = 2, \quad \mu = (r - \sigma^2/2)(T - t), \quad \text{and } s^2 = \sigma^2(T - t).
\]

(If you’re really rigorous, you have to refer to “the useful rule” to ensure that the \(F_t\)-measurable, but stochastic, object \(S(t)\) in the indicator function can in fact be treated as a constant.) This gives

\[
e^{2(r - \sigma^2/2)(T-t) + 2\sigma^2(T-t)} \left(1 - \Phi \left( \frac{\ln(K/S(t)) - (r - \sigma^2/2)(T - t) - 2\sigma^2(T - t)}{\sigma \sqrt{T-t}} \right) \right)
\]

Recalling the \(1 - \Phi(x) = \Phi(-x)\) (the \(N(0,1)\)-distribution is symmetric around 0) & collecting the terms gives the desired result.

**EXERCISE 3; TERM STRUCTURE & CHANGE OF NUMERAIRE**

3a [10%]

We have

\[
L_\delta(t; T_i) = \frac{1}{\delta} \left( \frac{P(t, T_i)}{P(t, T_{i+1})} - 1 \right).
\]

Note that \(P(t, T_i)/P(t, T_{i+1})\) is the price of a traded asset divided by \(P(t, T_{i+1})\). So it’s a martingale under \(Q^{T_{i+1}}\) (by Sætning 5 in the “Numeraire”-note or Theorem 19.8 in Björk). Adding a constant & multiplying by another one doesn’t ruin the martingale property. So \(L_\delta(t; T_i)\) is a \(Q^{T_{i+1}}\)-martingale. (We don’t need Ito & Girsanov, but of course using those to calculate \(dL_\delta(t; T_i)\) & seeing that drift terms vanish under \(Q^{T_{i+1}}\) is completely OK. And so is rewriting to get “our favorite way of making martingales “\(E_t(Y)\)”.

3b [10%]

By definition of the caplet & the \(T_{i+1}\)-forward measure we have

\[
\pi^{\text{caplet}}(0; T_i, \delta, \kappa) = P(0, T_{i+1})E_t^{Q^{T_{i+1}}} \left( \delta(L_\delta(T_i; T_i) - \kappa)^+ \right).
\]

We also have that \(L_\delta(t; T_i)\) is a 0-drift Geometric Brownian motion under \(T_{i+1}\)-forward measure, so

\[
L_\delta(T_i; T_i) = L_\delta(0; T_i) \exp \left( -\frac{1}{2} \gamma^2(T_i) T_i + \gamma(T_i) W^{Q^{T_{i+1}}}(T_i) \right)
\]
The formula follows by the standard calculation of expected values of truncated log-normal variables.

3c. [20%]

The time-\(T_i\) cash-flow from the floating rate annuity is

\[ A(T_i) + \delta H(T_{i-1}) * L_\delta(T_{i-1}; T_{i-1}) \]

Note that \(H(T_{i-1}) = \sum_{j=1}^{N} A(T_j)\). The \(A\)’s can be calculated explicitly with annuity formulas (involving \(\alpha_{n}\), as in for instance Section 3.3 in the I&F-theory-notes) — but there’s no need to.

Consider a portfolio of of \(N\) bullets, specifically \(A(T_i)\) (principal-1) \(T_i\) bullets for each \(T_i\) (such a portfolio can be formed at time 0 since the \(A(T_i)\)’s are known at time 0). The time-\(T_i\) cash-flow from this portfolio:

\[ A(T_i) + \delta L_\delta(T_{i-1}; T_{i-1}) \sum_{j=1}^{N} A(T_j) = A(T_i) + \delta L_\delta(T_{i-1}; T_{i-1}) H(T_{i-1}). \]

So the future cash-flows of the floating rate annuity & the bullet portfolio are the same, hence there \(T_0\)-value must be the same. From Björk page 237 we know that the value of \(A(T_i)\) \(T_i\)-bullets is simply \(A(T_i)\). So the value of the total portfolio is \(\sum_{j=1}^{N} A(T_i) = H(0)\). (The same can be shown with “high-powered” \(\mathbb{E}^Q/\mathbb{E}^Q_\pi\)-calculations (combined with \(H(T_i) = H(T_{i-1}) - A(T_i)\), and that’s of course perfectly OK, but “overkill”, really.)

Any loan type with an instalment plan fixed at time 0 will satisfy \(H(T_{i-1}) = \sum_{j=1}^{N} A(T_j)\), and the exact same argument as above shows that its time-0 value is \(H(0)\) when the borrower is paying LIBOR.

The time-\(T_i\) cash-flow from the “HusY”-loan can be written as

\[ A(T_i) + \delta H(T_{i-1}) * \min(L_\delta(T_{i-1}; T_{i-1}), \kappa) \]

\[ = A(T_i) + \delta H(T_{i-1}) * L_\delta(T_{i-1}; T_{i-1}) - \delta H(T_{i-1})(L_\delta(T_{i-1}; T_{i-1}) - \kappa)^+ \]

i.e. as the cash-flow from an uncapped loan minus the cash-flow of \(H(T_{i-1})\) caplets.

The value of the all cash-flows from the “HusY”-loan the has a time \(T_0\)-value of

\[ H(0) - \sum_{i=1}^{N} H(T_{i-1}) \pi_{caplet}(0; T_{i-1}, \delta, \kappa). \]

With the caplet price formula from above, this can be calculated explicitly.

The time-\(T_i\) cash-flow from the “SmartHusY”-loan can be written as

\[ A(T_i) + \delta H(T_{i-1}) * \min(L_\delta(T_{i-1}; T_{i-1}) + s, \kappa + s) \]

\[ = A(T_i) + \delta H(T_{i-1}) * \min(L_\delta(T_{i-1}; T_{i-1}), \kappa) + \delta H(T_{i-1}) s \]
Rewriting again in terms of a caplet & summing over \( T_i \)'s we find that for a given \( s \) the value of a “SMARTHUSY”-loan is

\[
H(0) - \sum_{i=1}^{N} H(T_{i-1})\pi^{\text{caplet}}(0; T_{i-1}, \delta, \kappa) + \delta H(T_{i-1}) P(0, T_i) s
\]

\[
= H(0) - \sum_{i=1}^{N} H(T_{i-1}) (\pi^{\text{caplet}}(0; T_{i-1}, \delta, \kappa) - P(0, T_i) \delta s).
\]

So to find \( s \) we have to solve

\[
\sum_{i=1}^{N} H(T_{i-1}) (\pi^{\text{caplet}}(0; T_{i-1}, \delta, \kappa) - P(0, T_i) \delta s) = 0
\]

and that’s easy:

\[
s = \frac{\sum_{i=1}^{N} H(T_{i-1})\pi^{\text{caplet}}(\ldots)}{\sum_{i=1}^{N} \delta H(T_{i-1}) P(0, T_i)}
\]

All other things being equal, the “SMARTHUSY”-loan is tractable for the private home-owners because the price of the caplets is “transferred” to interest payments that are tax deductible. Of course that only works if tax authorities “buy” this construction, but apparently they do in Denmark. Further, one could ask if the diminished payments (after tax) for the borrower aren’t exactly off-set by the investor receiving less (after tax), so in equilibrium, it wouldn’t matter? Fair point, but investor/borrower tax asymmetries may/will still create a “net gain” from the construction.