The Ito formula

- Box-calculus: $dW \cdot dW = dt$, $dt \cdot dW = 0$, etc.
- 1-dim: With $Y(t) = f(X(t))$ then
  \[ dY = f'(X) dX + \frac{1}{2} f''(X) (dX)^2 \]
- Product rule:
  \[ d(XY) = Y dX + X dY + dXdY \]
- Otherwise look it up ...

**Mathematical Finance**

**PRICING BY "NO ARBITRAGE" IN CONTINUOUS TIME**

Binomial models: Don’t (ever) forget them.

Martingales

Brownian motion: Building-block for cont'-time martingales

Stochastic integrals: $\lim_{n \to \infty} \sum_{i=1}^{n} g(t_{i-1}) \Delta W(t_i)$

Stochastic integrals are martingales.

Ito-processes: $dX(t) = \mu(t) dt + \sigma(t) dW$

Ito-processes are martingales $\Leftrightarrow \mu(t) \equiv 0$.

Ito-processes as models of continuous time financial prices

(Self-financing) portfolios: Stochastic integrals “the right tool”

Self-financing condition: $dV^\phi = \phi^T dS$, where $V^\phi = \phi^T S$

An arbitrage is “something for nothing” / “a free lunch”. Too good to be true. So there aren’t any. (Economic reasoning. Very weak requirement.)

How to spot an arbitrage: $dV^\phi = \mu(t) V^\phi dt$, where $\mu(t) \not\equiv r$.

Feynman-Kac: Link between PDEs and SDEs.

Girsanov: Equivalent measure changes alters drift, not volatility.

Geometric Brownian motion

\[ dX = \alpha X dt + \sigma X dW \]

\[ X(t) = X(s) \exp((\alpha - \sigma^2/2)(t-s) + \sigma(W(t) - W(s))) \]

Ornstein-Uhlenbeck: $dX = (\alpha + \beta X) dt + \sigma dW$

For deterministic $h$-fct.: $\int_0^1 h(u) dW(u) \sim N(0, \int_0^1 h^2(u) du)$
Feynman-Kac: Sol’n can be represented via mean of “artificial process”.

Girsanov: By changing to the probability measure \( Q \) we may use \( S \) itself:

\[
\pi^{\text{call}}(t) = e^{-r(T-t)} E^Q_t \left((S(T) - K)^+\right)
\]

where

\[dS = rSdt + \sigma SdW^Q\]

Note: \( \beta/\beta \), \( S/\beta \) and \( \pi/\beta \) are all \( Q \)-martingales; \( Q \) is an equivalent martingale measure.

"Base-case" Black-Scholes model

\[
d\beta(t) = r\beta(t)dt \\
dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \Rightarrow \frac{dS}{S} = \mu dt + \sigma dW
\]

Simple contingent claim, call-option for instance.

\( \pi^{\text{call}}(T) = (S(T) - K)^+ \), but what is \( \pi^{\text{call}}(t) \) for \( t < T \) ?

\( \pi^{\text{call}}(t) = F(S(t), t) \) seems reasonable/necessary ("a Markov-feeling")

Ito-formula + smart (stock,call)-pf. + no arbitrage \( \Rightarrow \) Fund. PDE ("with no \( \mu \)"

\[
F_t + rxF_x + \frac{1}{2}\sigma^2 x^2 F_{xx} = rF \quad F(x, T) = (x - K)^+
\]

Martingale formulation: More general.

1st Fund. Th. of Asset Pricing: A model is arbitrage-free if and only if an equivalent martingale measure \( Q \) exists. This means that

\[
\pi(t) = \beta(t) E^Q_t \left( \frac{\pi(T)}{\beta(T)} \right) \Rightarrow \frac{d\pi}{\pi} = rdt + \sigma dW^Q.
\]

Completeness: Any claim (also path-dep. ones) can be replicated.

2nd Fund. Th. of Asset Pricing: An arbitrage-free model is complete if and only if \( Q \) is unique.

1st & 2nd FToAP: OK under technical conditions.

Direct calculation \( \rightarrow \) Black-Scholes formula. Involves \( E(X1_{X \geq 0}) \) where \( X \sim \log N \). Doable. A general result in an exercise.

Patching up & extending:

The argument works for all simple claims.

A simple claim be replicated with a (bank-account, stock)-pf.; a hedge-pf. or replicating-pf. This means holding \( \Delta(t) = F_x(S(t), t) \) \( (= \Phi(d_1) \) for the call) units of stock.

The PDE approach works for \( \sigma = \sigma(S(t), t), \mu \) general, and in \( n \) dimensions.
BIG TOPIC: BOND MARKET MODELS

Definitions (many different types of interest rates)

Prop. 15.5: Link between objects. “Pure algebra”, “measure independent”. Very useful.

Just specifying $\mathbb{P}$-dynamics of $r$ isn’t enough. Risk-premia are needed. “But almost”.

Affine short rate models: ZCB-prices of the form $P(t,T) = \exp(A(t,T) - B(t,T)r(t))$. $A$ & $B$ solve ODEs.

SMART TECHNIQUE: CHANGE OF NUMERAIRE

Idea: For each traded asset $S$ existence of a measure $Q^S$ such that $\pi/S$ is a $Q^S$-martingale is equivalent to no arbitrage,

Formula:

$$\pi(t) = \beta(t)E_t^{Q^S}\left(\frac{\pi(T)}{\beta(T)}\right) = S(t)E_t^{Q^S}\left(\frac{\pi(T)}{S(T)}\right)$$

Girsanov identifies the $Q^S$-BM and hence $Q^S$-dynamics.

$T$-ZCB as numeraire: The forward measure

The affine models: CIR and Vasicek. Solved explicitly.

Calibration: Hull/White (ext. Vasicek) worked through.

1st FToAP. + Prop. 15.5 gives HJM-drift restriction:

$$\alpha^Q_f(t,T) = \sigma_f(t,T) \int_t^T \sigma_f(t,u)du$$

Girsanov can be used to show completeness.

The (extended) B/S-model is complete.

More generally: $\dim(W) = \#$ (“exogenously given”) traded assets means completeness. More Brownian motions than that: Incompleteness.


Dividend application: Multi-currency models.
Applications/uses of change of numeraire

- B/S formula without an \( i \)-sign & the exchange option easily
- A “dimension \( d \)” call-option formula in lognormal models
- Formula for ZCB call-option prices in ext. Vasicek (hard without).
- Caps & (Lognormal) LIBOR market models
- A formula for coupon-bond call option prices in the ext. Vasicek (works in dim. 1 “only”)

SOME OF WHAT WE DIDN’T DO

- Treat stochastic integrals completely rigorously.
- Explore technical conditions for 1st & 2nd FToAP.
- Estimate parameters “properly”. little empirics at all, in fact.
- Computational finance is largely unexplored.
- Price exotics (American, Asian, Russian, Parisian, …)

But you’re ready for it!

Some topics are treated in other courses. [slides with courses]

Some aren’t!

Fortunately it’s a free country, so you are allowed to read for yourself. And we have projects & theses.

Applications/uses of change of numeraire

- Consider market frictions (transactions costs, short-selling constraints, liquidity)
- \textit{Really} go beyond Black-Scholes
  - Implied/local volatility models (complete)
  - Stochastic volatility models (“mildly” incomplete)
  - Jump-diffusion model (“very” incomplete)
  - “Other Levy Process”-driven stock-prices (“grossly” incomplete)
This exam is exactly as individual as a standard 4-hour written one. Any group-work is cheating, and will be dealt with with great vengeance. Questions: Ask me. (Evidently, the are some questions I won’t answer.)

Answer in Danish or English as you see fit. I can read my own handwriting, so I can almost certainly read yours. Use any result that is in the book, but give a precise reference.

"Hard" numerical work: No. Numbers: Maybe.

If you get stuck, go on. Questions are fairly independent, their weights clearly marked, and errors (as far as possible) only punished once.

THE EXAM

Curriculum: Øksendal 8.F (Girsanov) + Björk, but no questions that require knowledge of Chapters 13,14 and 20.

Question session: Monday January 5; 10-12 in ???

Exam-questions: Handed out (by me in my office; E-411) at 9 am on Tuesday January 6. Also posted on the homepage.

Answers: Hand no later than 12 noon on Wednesday January 7. Hand in to me or to the secretary Mette on the 3rd floor. TWO COPIES PLEASE. Electronic hand-in is OK, but at your own risk.

Results: Posted (by exam-number) on the homepage.