1 Exercise 4.3 in Björk

Start by making sure that you do understand the proof for exercise 4.1 in Björk (as done in the classes) and that Ito can be used on real-valued mappings only. The following is merely a lot of indices.

Proposition 4.3 states that the $n$-dimensional linear sde:

$$\begin{cases}
  dX_t = (AX_t + b_t)dt + \sigma_t dW_t \\
  X_0 = x_0
\end{cases}$$

where $A \in \mathcal{M}(n,n)$, $b \in \mathbb{R}^n$ and $\sigma \in \mathcal{M}(n,d)$, has a solution given by

$$X_t = e^{At}x_0 + \int_0^t e^{A(t-s)}b_s ds + \int_0^t e^{A(t-s)}\sigma_s dW_s$$

the SDE can be written in the following manner:

$$d\begin{pmatrix}
  X_{1,t} \\
  \vdots \\
  X_{n,t}
\end{pmatrix} = \begin{pmatrix}
  b_{1,t} + \sum_{j=1}^n a_{1,j}X_{j,t} \\
  \vdots \\
  b_{n,t} + \sum_{j=1}^n a_{n,j}X_{j,t}
\end{pmatrix} dt + \begin{pmatrix}
  \sigma_{11} & \cdots & \sigma_{1d} \\
  \vdots & \ddots & \vdots \\
  \sigma_{n1} & \cdots & \sigma_{nd}
\end{pmatrix} d\begin{pmatrix}
  W_{1,t} \\
  \vdots \\
  W_{n,t}
\end{pmatrix}$$

$$= \begin{pmatrix}
  b_{1,t} + a_{1,1}X_t \\
  \vdots \\
  b_{n,t} + a_{n,1}X_t
\end{pmatrix} dt + \begin{pmatrix}
  \sigma_1 \\
  \vdots \\
  \sigma_n
\end{pmatrix} dW_t$$

where $a_j$ and $\sigma_j$ represent row number $j$ in $A$ respectively $\sigma$. Thus

$$dX_j = b_j + a_jX_t dt + \sigma_j dW_t$$

Consider

$$Z_t = e^{-At}X_t \equiv M_t X_t$$

i.e.

$$Z_t = \begin{pmatrix}
  M_1X_t \\
  \vdots \\
  M_nX_t
\end{pmatrix}$$

1
where again $M_j$ is an entire row. Notice also that (ex. 4.2 in Björk)

$$dM_t = -AM_t dt = - \left( \begin{array}{ccc} a_1 M^1 & \ldots & a_1 M^n \\ \vdots & \ddots & \vdots \\ a_n M^1 & \ldots & a_n M^n \end{array} \right) dt$$

where $M^j$ is the number $j$ column in $M$. Thus

$$dM_j = -[a_j M^1, \ldots, a_j M^n] dt$$

All of the above give us that

$$dZ_j = X_t dM_j + M_j dX_t + dM_j dX_t$$

$$= -[a_j M^1, \ldots, a_j M^n] dt X_t + M_j (AX_t + b) dt + M_j \sigma dW_t$$

$$= -[a_j M^1, \ldots, a_j M^n] dt X_t + [M_j a^1, \ldots, M_j a^n] X_t dt + M_j b dt + M_j \sigma dW_t$$

now realize that in $n$-dimensions the above can be written as

$$dZ = -AX dt + M b dt + M \sigma dW_t$$

the right handside doesn’t depend on $Z$, so we can simply integrate in order to obtain $Z$

$$Z_t = e^{-At} X_t = Z(0) + \int_0^t e^{-A(t-s)} b_s ds + \int_0^t e^{-A(t-s)} \sigma_s dW_s$$

with $Z(0) = I \cdot X(0) = x_0$ that is

$$X_t = e^{-At} x_0 + \int_0^t e^{-A(t-s)} b_s ds + \int_0^t e^{-A(t-s)} \sigma_s dW_s$$

Please note that we had to look at $Z_{j,t}$ because Ito works on real-valued functions only.