Extra Exercise: Conditional Expectations

Consider a finite probability space \((\Omega, \mathcal{F}, P)\) on which \(X\) and \(Y\) are real-valued random variables. For \(x_k \in \mathbb{R}\) and some event \(A\) (ie. \(A \in \mathcal{F}\)), the "conditional probability \(A\) given \(X = x_k\)" is defined by

\[P(A|X = x_k) = \frac{P(A \cap \{\omega | X(\omega) = x_k\})}{P(\{\omega | X(\omega) = x_k\})} = \frac{P(A, X = x_k)}{P(X = x_k)},\]

when \(P(X = x_k) > 0\); otherwise it is by definition 0.

The conditional mean of \(Y\) given \(X = x_k\) is defined by

\[E(Y|X = x_k) = \sum_{\omega} Y(\omega)P(\omega|X = x_k).\]

These are definitions that appear in introductory probability courses. Conditional expectations given some sigma-algebra \(\mathcal{H}\) are not usually encountered. You should look up the definition of this; for instance in the lecture notes from "Investments and Finance Theory".

To get down to business, look at \(\Omega = \{\omega_1, \omega_2, \ldots, \omega_8\}\). Assume that \(P(\omega_i) = \frac{1}{8}\) for all \(i\). Let

\[X(\omega_i) = i \text{ for } i = 1, \ldots, 8.\]

and let

\[
\begin{align*}
Y(\omega_1) &= Y(\omega_2) = 1 \\
Y(\omega_3) &= Y(\omega_4) = 3 \\
Y(\omega_5) &= Y(\omega_6) = 5 \\
Y(\omega_7) &= Y(\omega_8) = 7
\end{align*}
\]

What is the smallest sigma-algebra \(\sigma(Y)\) that makes \(Y\) measurable? What is \(E(X|\sigma(Y)) := E(X|Y)\)? What is \(E(X|Y = 3)\) & how is it related to (for instance) \(E(X|Y)(\omega_i)\)? What is \(E(Y|X)\)?