Non-exotic Options

Suppose that $X$ is normally distributed, $X \sim N(\mu, \sigma^2)$, and $a$, $l$, and $h$ ($\geq l$) are real numbers. Show (using for instance the same ideas as when we derived the Black-Scholes formula) that

$$E \left( e^{aX} 1_{l \leq X \leq h} \right) = e^{\mu + \frac{a^2}{2}} \Phi \left( \frac{h - (\mu + a\sigma^2)}{\sigma} \right) - \Phi \left( \frac{l - (\mu + a\sigma^2)}{\sigma} \right),$$

where as usual $\Phi(\cdot)$ denotes the distribution function of the standard normal distribution. Do $l$ and $h$ really have to be finite?

Now consider the base-case lognormal Black-Scholes model, i.e.

$$dS(t) = \alpha S(t) dt + \sigma S(t) dW^P(t),$$
$$d\beta(t) = r\beta(t) dt.$$

Use the formula derived above to find time $t$ prices of the following simple financial contracts. (In reality: Do as many as you need to get the point!)

- The **binary option**, which has pay-off $1_{K \leq S(T) \leq K'}$.
- The **cash-or-nothing option** pay-off $K1_{S(T) > K}$.
- The **the gap option** which has pay-off $(S(T) - K)1_{S(T) > K}$ (Note there is no $()^+$ and no requirement that one $K$ is bigger than the other. The case $K = 0$ gives the so-called asset-or-nothing option.)
- The **power option** which has pay-off $(S^p(T) - K)^+$, for some $p \in \mathbb{R}$.