Note that

The course has ended. Or the lectures have, at least, i.e. no lectures on Monday December 17 or Tuesday December 18.

There are exercises on Thursday December 20.

Exam Info

I’ll make the questions public (on the homepage) at 9 am (9:00) on Thursday January 3, 2002. Of course you can also show up “in person” at my office and get them.

Answers must be handed in no later than 3 pm (15:00) on Monday January 7, 2002.

“Handed in” means i) given directly “in hard-copy” to me (hand-written answers are of course fine) or, ii) put in the mail-box outside the Mat-K-secretariat on the 4th floor in the E-building or, iii) delivered electronically to me in some recognized & printable file format. I don’t mind electronic versions (it does seem to indicate “too much time on your hands”, but …). However, I am not in any way responsible for computer failures. Don’t forget to put your name and/or exam-number on your answers. My lists contain both pieces of info, so one is sufficient & anyway you’re not anonymous.

It’s an individual exam, no working in groups! So if you have questions, please ask me. (Obviously, there are many questions I will not answer.)

Answers will be graded on the 13-scale.

Given a proper reference, you’re allowed to use any result that you can find in the book, the notes, or the exercises “without further ado”. In fact, you’re encouraged to.

The individual questions & their weights will be clearly marked. Two questions both being marked “10%”, does not make them equally difficult. Questions can be answered fairly independently, so if you get stuck, go on.

Past lectures

Calculations with the change of numeraire technique.
We saw how the price of the exchange option could be found. We already found the price in Chapter 7, but this time it was considerably easier. The $T$-forward measure, $Q^T$ was introduced. This is the martingale measure corresponding to the $T$-ZCB, $P(\cdot, T)$ as numeraire. Since $P(T, T) = 1$ we immediately have that arbitrage-free prices can be written as

$$\pi(t) = P(t, T)E_{Q^T}^t (\pi(T)).$$

If interest rates are deterministic, then $\sigma_r(t, T) = 0$ (the ZCB-volatility, as defined in Chapter 15), and $Q = Q^T$. We derived a general call-option price formula. In the lognormal case (Assumption 19.5.1) we derived an explicit, Black-Scholes-like version of the formula, sometimes called the Geman-El Karoui-Rochet formula (Should you ever talk to Farshid Jamshidian, using this name isn’t advisable. The idea of change of numeraire was developed independently and simultaneously, it appears, by Jamshidian & French probabilist Helyette Geman, each of whom claims to have “known about it for some time”).

Tuesday December 4: The Rest of Björk.
Using the general call-option formula for lognormal discounted processes we derived an explicit price formula (Proposition 19.6) for options on ZCBs the the Hull-White (extended Vasicek) model. This formula would be very difficult to derive without the change of numeraire technique. Note also how the the calibration parameter/function $\Theta$ enters the formula only through current ZCB-prices $P(t, T_1)$ and $P(0, T_2)$. So you don’t have to actually do the calibration, it’s enough to say you did! Further, this way of determining call-prices for ZCB-options works perfectly well multi-dimensional lognormal models.

We looked a little at the unbiased expectations hypothesis,

$$f(t, T) = E_{Q^T}^t (r(T)).$$

We saw that if “$\pi$” is understood as $Q^T$ then it’s true (but then there’s a different expectation for each date $T$). If “$\pi$” is taken to mean $P$ or $Q$ (remembering the definition/interpretation of forward rates & $Q$ as “as if risk-neutrality” especially the latter might seem reasonable), then the hypothesis will either only hold in very “carefully constructed models” (certainly none we will encounter; dimension $\geq 2$) or be plain nonsense (dimension 1). The fact that $E_{Q^T}^t (r(T)) = f(t, T)$ lead to a general pricing formula that could easily come in handy (Proposition 19.17).

In Chapter 20 we looked at forward and futures contracts. Forward contracts are simple enough. Futures are somewhat more involved. They can be thought
of as “forwards that are settled along the way”. Technically, its an asset whose price is always 0, and has a dividend process \( F(t; T, S) \) (\( S \sim \text{underlying} \)) such that \( F(T, T, S) = S(T) \). Futures price is “bad lingo”; the futures price isn’t technically a price, but a cumulative dividend process. Anyway, with the definition in place, we quite easily got that
\[
F(t; T, S) = E^{Q}(S(T)),
\]
so the futures price (not its discounted version!) is a \( Q \)-martingale If interest rates are independent of the underlying, then the forward and the futures prices are the same (but that doesn’t make them the same contract).

**Proof** Since \( \pi^{FUT} \equiv 0 \), \( V^{\pi}(t) = \phi_{h}(t)\beta(t) + \phi^{FUT}(t)\pi^{FUT}(t) = \pi^{\pi}(t) \). To show that the strategy is self-financing we have the show that
\[
dV^{\pi}(t) = (\phi_{h}(t), \phi^{FUT}(t))^\top dG^{FUT},
\]
where \( G \) is gains processes, i.e. price + cumulative dividends, see Chapter 5 in Björk. The left hand side of (1) is
\[
dV^{\pi}(t) = d\pi^{\pi}(t)(MG-propr. \& \text{Int}) \pi^{\pi}(t) dt + \pi^{\pi}(t) S(t) dW^{Q}(t)
\]
For the bank-account we have \( dG^{Q} = \beta(t) dt \), while
\[
dG^{FUT}(t) = \frac{d(0 + F(t; T, S))}{dF(t; T, S)} = dF(t; T, S) = d(\phi_{h}(t) e^{rT} S(t)) = \sigma e^{rT} dt + \sigma \Delta(t) S(t) dW^{Q}(t)\]
(Note that we don’t have to spend time calculating the \( Q \)-drift of the futures price; we know it’s 0 because we’re dealing with a martingale. Of course, you can check if you like.) Therefore by insertion we get the right hand side of (1)
\[
\pi^{\pi}(t) \frac{\beta(t)}{\beta(t)} dt + \Delta(t) e^{r(T-t)} S(t) dW^{Q}(t) = \pi^{\pi}(t) dt + \sigma \Delta(t) S(t) dW^{Q}(t),
\]
which was the left hand side, and we’re done.

I talked about caplets. These are “options on interest rates”, in the sense that they pay off (floating-rate - fixed rate)\( ^{+} \) at one (future) point in time. (And a cap is a portfolio of caplets.) Following Björk’s Section 19.3, we saw that a caplet on the LIBOR rate is equivalent to a put option on a ZCB. So in the Hull/White (extended Vasicek) model we’d know how to price them. Then I talked about swaptions. This is the right (but not the obligation) to enter into a (specific) swap contact with a (specific) fixed rate, say \( \kappa \), at a (specific) future date. Go back to Björk’s Section 15.3.3 if you’ve forgotten what a swap contract (and a swap-rate) is. Graphically, things look like this:
At time $T_l$, the value of the swaption is:

$$\delta(\omega(T_l; T_m, T_n, \delta) - \kappa)^+ \sum_{j=m+1}^n P(T_j, T_l),$$

where $\omega(T_l; T_m, T_n, \delta)$ is the time-$T_l$ swap rate for the underlying swap (sometimes called the forward par swap-rate). Recall that this is the value of the fixed rate that gives the swap-contract a market value of 0. We have from Section 15.3.3 that

$$\omega(T_l; T_m, T_n, \delta) = P(T_l; T_m) - P(T_l; T_n) \frac{\delta n \sum_{j=m+1}^n P(T_j, T_l)}{\delta \omega}$$

Why is the swaption pay-off that? Well, if $\kappa > \omega$, we’ll never enter into a $\kappa$-swap, for a price of 0 we can get one where we’re paying a lower fixed rate ($\omega$). If on the other hand $\kappa < \omega$, we enter into the “fixed rate $\kappa$"-swap (that’s free, remember). The floating rate payments net out, and at date $T_j$ ($m + 1 \leq j \leq n$) we receive $\delta \omega$ and pay $\delta \kappa$. All of these payments are known at time $T_l$ and thus we get the time-$T_l$ swaption value. So the time-$t$ price of the swaption is

$$\pi_{\text{swopt}}(t; T_l, T_m, \delta, \kappa) = P(t, T_l) \mathbb{E}_t^Q \left( \delta(\omega(T_l; T_m, T_n, \delta) - \kappa)^+ \sum_{j=m+1}^n P(T_j, T_l) \right)$$

Note that all 5 “time variables” ($t, T_l, T_m, T_n$, and $\delta$) have to be specified before we know exactly which swaption price we’re talking about. (5 time indices; that’s a “course high”.) Just saying “the swaption price” leaves something to the reader. Assuming (WLOG, for our purposes) that $T_l = T_m$ we can rewrite the swaption pay-off as

$$\left(1 - \sum_{j=m+1}^n \alpha_j P(T_j, T_l) \right)^{\omega},$$

with $\alpha_j = \delta \kappa$ for $j \leq n - 1$ and $\alpha_n = 1 + \delta \kappa$. So we see that a swaption is similar to a put-option on a coupon-bearing bond.

Tuesday November 11: Options on coupon bonds. Implied volatility.
We took a look at how to price call-options on coupon-bearing bonds. (Clearly it works the same way for puts.) That’s important because i) real-world bonds have coupons, ii) it’s a non-trivial extension of ZCB option pricing, iii) it’s a nice application for change of numeraire techniques, and iv) the approach could prove useful in other contexts. The theory is described in a separate note, that will appear on the homepage shortly (after I’ve written it).

I mentioned implied volatility (see Björk Section 6.7.2). Traders & the like often quote option prices in terms of the implied volatilities. (Implied from some, typically Black-Scholes-like, formula that they may or may not understand/believe in.)

Thursday November 13: Guest from Nykredit. Brief summary of the “slids” & “didn’ts” in the course.
Jesper Lund and Peter Honoré talked about mortgage-backed securities and interest rate derivatives (caps & swaptions). Their slides can be found on the homepage.

I closed up the course; my slides, too, can be found on the homepage.

Coming lectures
There aren’t any!

Exercises for Week 51 (Thursday December 20; no hand-in.)

Homemade Exercise 8.1 (An old exam question.)

Homemade Exercise 8.2 (Options on futures.)

Björk Exercises 19.1, 19.2 (integrate ‘till you’re blue in the face), 19.3 (here “equivalent” means “has the same value at any time $t \leq T_{i-1}$ irrespective of the dynamic model”)

Kindly,

Rolf