Mathematical Finance

PRICING BY "NO ARBITRAGE" IN CONTINUOUS TIME

Binomial models: Don’t (ever) forget them.

Martingales

Brownian motion: Building-block for cont’-time martingales

Stochastic integrals: 

\[ \lim_{n \to \infty} \sum_{i=1}^{n} g(t_{i-1}) \Delta W(t_{i}) \]

Stochastic integrals are martingales.

Ito-processes: 

\[ dX(t) = \mu(t)dt + \sigma(t)dW \]

Ito-processes are martingales \( \Leftrightarrow \mu(t) \equiv 0 \).

Ito-processes as models of continuous time financial prices

(Self-financing) portfolios: Stochastic integrals “the right tool”

Self-financing condition: 

\[ dV^\phi = \phi^T dS, \quad \text{where} \quad V^\phi = \phi^T S \]

An arbitrage is “something for nothing”/“a free lunch”. Too good to be true. So there aren’t any. (Economic reasoning. Very weak requirement.)

How to spot an arbitrage: 

\[ dV^\phi = \mu(t)V^\phi dt, \quad \text{where} \quad \mu(t) \not\equiv r. \]

Feynman-Kac: Link between PDEs and SDEs.

Girsanov: Equivalent measure changes alters drift, not volatility.

Geometric Brownian motion

\[
\begin{align*}
    dX &= \alpha X dt + \sigma X dW \\
    \Upsilon &\to X(t) = X(s) \exp((\alpha - \sigma^2/2)(t-s) + \sigma(W(t) - W(s)))
\end{align*}
\]

Ornstein-Uhlenbeck: 

\[ dX = (\alpha + \beta X)dt + \sigma dW \]

For deterministic \( h \)-fct.:

\[ \int_0^t h(u)dW(u) \sim N(0, \int_0^t h^2(u)du) \]
Feynman-Kac: Sol’n can be represented via mean of “artificial process”.

Girsanov: By changing to the probability measure $Q$ we may use $S$ itself:

$$\pi^{\text{call}}(t) = e^{-r(T-t)}E^Q_t \left( (S(T) - K)^+ \right)$$

where

$$dS = rSdt + \sigma SdW^Q$$

Note: $\beta/\beta$, $S/\beta$ and $\pi/\beta$ are all $Q$-martingales; $Q$ is an equivalent martingale measure.

"Base-case" Black-Scholes model

$$d\beta(t) = r\beta(t)dt$$

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \Leftrightarrow \frac{dS}{S} = \mu dt + \sigma dW$$

Simple contingent claim, call-option for instance.

$$\pi^{\text{call}}(T) = (S(T) - K)^+$$

but what is $\pi^{\text{call}}(t)$ for $t < T$?

$$\pi^{\text{call}}(t) = F(S(t), t)$$ seems reasonable/necessary (“a Markov-feeling”)

Ito-formula + smart (stock, call)-pf. + no arbitrage $\Rightarrow$ Fund. PDE (“with no $\mu$”)

$$F_t + rxF_x + \frac{1}{2}\sigma^2 x^2 F_{xx} = rF \quad F(x, T) = (x - K)^+$$

Martingale formulation: More general.

1st Fund. Th. of Asset Pricing: A model is arbitrage-free if and only if an equivalent martingale measure $Q$ exists. This means that

$$\pi(t) = \beta(t)E^Q_t \left( \frac{\pi(T)}{\beta(T)} \right) \Rightarrow \frac{d\pi}{\pi} = rdt + \sigma dW^Q.$$  

Completeness: Any claim (also path-dep. ones) can be replicated.

2nd Fund. Th. of Asset Pricing: An arbitrage-free model is complete if and only if $Q$ is unique.

1st & 2nd FToAP: OK under technical conditions.

Direct calculation $\rightarrow$ Black-Scholes formula. Involves $E(X1_{X \geq 0})$ where $X \sim \log N$. Doable. HE 4.1 gives general result.

Patching up & extending:

The argument works for all simple claims.

A simple claim be replicated with a (bank-account, stock)-pf.; a hedge-pf. or replicating-pf. This means holding $\Delta(t) = F_x(S(t), t)$ ($= \Phi(d_1)$ for the call) units of stock.

The PDE approach works for $\sigma = \sigma(S(t), t)$, $\mu$ general, and in $n$ dimensions.


**BIG TOPIC: BOND MARKET MODELS**

Definitions (many different types of interest rates)

Prop. 15.5: Link between objects. “Pure algebra”, “measure independent”. Very useful.

Just specifying $\mathbb{P}$-dynamics of $r$ isn’t enough. Risk-premia are needed. “But almost”.

Affine short rate models: ZCB-prices of the form $P(t,T) = \exp(A(t,T) - B(t,T)r(t))$. $A$ & $B$ solve ODEs.

The (extended) B/S-model is complete.

More generally: $\dim(W) = \#$ (“exogenously given”) traded assets means completeness. More Brownian motions than that: Incompleteness.


Dividend application: Multi-currency models.

**SMART TECHNIQUE: CHANGE OF NUMERAIRE**

Idea: For each traded asset $S$ existence of a measure $Q^S$ such that $\pi/S$ is a $Q^S$-martingale is equivalent to no arbitrage,

Formula:

$$\pi(t) = \beta(t)E^Q_t \left( \frac{\pi(T)}{\beta(T)} \right) = S(t)E^Q_t \left( \frac{\pi(T)}{S(T)} \right)$$

Girsanov identifies the $Q^S$-BM and hence $Q^S$-dynamics.

$T$-ZCB as numeraire: The forward measure

The affine models: CIR and Vasicek. Solved explicitly.

Calibration: Hull/White (ext. Vasicek) worked through.

1st FToAP. + Prop. 15.5 gives HJM-drift restriction:

$$\sigma_f^Q(t,T) = \sigma_f(t,T) \int_t^T \sigma_f(t,u) du$$
Applications/uses of change of numeraire

- B/S formula without an $j$-sign
- Pricing the exchange option easily
- A "dimension $d$" call-option formula in lognormal models
- Formula for ZCB call-option prices in ext. Vasicek (hard without).
- A formula for coupon-bond call option prices in the ext. Vasicek (works in dim. 1 “only”)

Some of what we didn’t do

- Be rigorous about stochastic integrals.
- Explore technical conditions for 1st & 2nd FToAP
- Discuss estimating parameters. No empirics at all, in fact.
- Look a numerics. Computational finance is unexplored.
- Price exotics (American, Asian, Russian, Parisian, Bermudan, …)

But you’re ready for it!

Some topics are treated in other courses.

Most aren’t!

Fortunately, you are allowed to read working papers, articles & books yourself. (And we have projects & theses.)

- Consider market frictions (transactions costs, short-selling constraints, liquidity)
- Really go beyond Black-Scholes
  - Implied/local volatility models (complete)
  - Stochastic volatility models (“mildly” incomplete)
  - Jump-diffusion model (“very” incomplete)
  - “Other Levy Process”-driven stock-prices (“grossly” incomplete)