Mathematical Finance, Fall 2001, Homemade Exercise 1.2

Change of numeraire in binomial models & a nice fact

Consider a 1-period binomial model and adapt the notation in Björk’s chapter 2. (Except: We only write q for his q, and use parentheses for time indices.) Let f be a not-too-pathological (whatever that means) function and consider a financial contract with terminal payoff f(S₁). Let C denote the arbitrage-free price of this contract. An alternative, but since B(0) = 1 quite trivial, way of writing Björk’s Prop. 2.25 is as

\[
\frac{C(0)}{B(0)} = \mathbb{E}^Q \left( \frac{C(1)}{B(1)} \right).
\]

Put \( \varphi = \frac{uq}{1 + R} \) and show that \( \varphi \in [0, 1] \) \( \Leftrightarrow \) q \( \in \) [0, 1]. Let \( Q' \) be the probability measure we get by using \( \varphi \) as binomial probability in an arbitrage-free model. Show that

\[
\frac{C(0)}{S(0)} = \mathbb{E}^{Q'} \left( \frac{C(1)}{S(1)} \right)
\]

and that

\[
\frac{B(0)}{S(0)} = \mathbb{E}^{Q'} \left( \frac{B(1)}{S(1)} \right).
\]

Convince someone that this also works on an n-period binomial model, i.e. that

\[
\frac{\pi(0)}{S(0)} = \mathbb{E}^{Q'} \left( \frac{\pi(T)}{S(T)} \right),
\]

where \( \pi \) denotes the price of any traded security (i.e. \( B, S, \) or \( C \)).

The interpretation/intuition/motivation is that there is nothing special about using the bank-account as numeraire, i.e., measuring all prices relative to the bank-account, when it comes to saying that "no arbitrage is equivalent to the existence of an equivalent martingale measure". We could just as well measure in units of the stock (What problems could arise if we measured relative to, say, a call-option price?) But note that we get a different measure, \( Q' \neq Q \).

From an economic point of view, this is hardly surprising, and it might not be clear what the benefit of doing it could possibly be. This is a cliff-hanger; in Björk’s chapter 19 we’ll see how change of numeraire can be a big help both in computations and in model construction.

Prove that the value process for a self-financing strategy is a \( Q' \)-martingale when discounted by the bank-account. Recall that by iterated expectations