Binomial models revisited & revamped

Consider a 5-period financial model with a stock and a bank-account. The interest rate is 0, the stock pays no dividends (during the 5 periods considered, to be precise) and has price dynamics given by the lattice shown below. (It is understood that moves appear in the lattice if and only if the have strictly positive probability.)

Argue that this model is arbitrage-free and complete, and find the martingale probabilities.

Now consider the pricing of a so-called knock-out call-option. This is a path-dependent option with the pay-off of a standard strike-$K$ call-option at expiry $T$ provided the stock price has not gone above (say in the ‘$\geq$’-sense) a certain barrier $B$ ($B \geq K$ with no real loss of generality) before time $T$. If the barrier has been hit, then the pay-off is 0. (This is for simplicity, in fact we might just as well work with a deterministic rebate $R$ being paid when the barrier is hit.) Draw the stock price process in tree-form (i.e. in non-recombining fashion) and find the arbitrage-free price at time 0, $\tilde{K}O(0)$, of a knock-out call-option with $T = 5$, $K = 70$, and $B = 120$. How does the number of calculations you have to perform depend on the number of time-periods?

A closer at the problem reveals that a considerable simplification can be made. What we are really interested in is the price, $\tilde{K}O(t)$, of the knock-out at some time $t$ given that was above at time $t - 1$, i.e. $S$ had not touched the barrier. (Why, actually? What do you conclude if you observe a market price of 0 for the knock-out?) Now look at $\tilde{K}O$ in the $S$-lattice. At time $T$, $\tilde{K}O$ is equal to the standard call. At the ‘$S = 120$’-nodes [C and D] we have $\tilde{K}O = 0$, and similarly at any node above $S = 120$. This looks as below:

What must be the value of $\tilde{K}O$ be at node A? And what is then $\tilde{K}O$ at node B? And how does this help us to fill in the ‘$??$’s and find $\tilde{K}O(0)$? What does the replicating strategy (in stock and bank-account) for the knock-out call look like? What is the advantage over the tree-method used earlier? (This technique may appear new, but really you have seen it before, namely when American options are priced in a lattice. Here you are in fact also finding the option price at time $t$ given the option hasn’t been exercised at $t - 1$ or earlier, although some books obscure this fact!)

How would you value a knock-in call-option (Hint: Take a wild guess of what ‘knock-in’ means and then think of the relation between knock-in, knock-out

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A third way of pricing and hedging has recently been considered, namely by so-called static hedging. The idea is to create a portfolio of time 0 that does not have to be dynamically readjusted (unless at boundaries; this is called unwinding) and replicates the pay-off of the knock-out call. Of course this cannot be done using the stock and bond alone, but in liquid markets it can be very reasonable to assume that standard (plain vanilla) calls and puts with a variety of strikes and expiry dates are traded (at their arbitrage-free prices) in the market. To be more specific, look at the following plan. To replicate a knock-out call, it seems natural to buy a standard call with same characteristics \((T = 5, K = 70)\). Make a lattice with the values of such a portfolio. This matches terminal pay-off (of the knock-out), but not the pay-off at the upper boundary (nodes \(C\) and \(D\)). Now consider adding to the portfolio a short position of \(n_C\) strike-120 expiry-5 call-options. What must \(n_C\) be in order to create a value of 0 at node \(C\)? What happens to the terminal nodes? What is the value of the resulting portfolio at all nodes in the lattice? Now consider a further addition to the portfolio, namely so many strike-120 expiry-3 calls, say \(n_D\), that a 0 is created at node \(D\). Find \(n_D\) and the portfolio value in all nodes. Argue that all in all you have created a portfolio that statically hedges the knock-out call. (What do you have to do at the boundaries?) Is the static hedge-portfolio that you’ve found unique (i.e. what about other strikes and maturities)?

As pointed out, static hedging is a fairly 'hot' in financial engineering in fact this exercise comes tantalizingly close plagiarism of the article Derman, Ergen & Kani (1995). The ideas of static hedging work also in continuous time, Carr, Ellis & Gupta (1998) is a standard reference, but we don’t have time to look at that in this course. It is, however, a topic that is very suitable for some kind of project.

References
