Mathematical Finance, Fall 2001, Homemade Exercise 2.1, Solution

After correcting for typos we have that

\[ X(t) = a(t) \left( x_0 + \int_0^t b(s) ds + \int_0^t c(s) dW(s) \right). \]

Note that Björk’s Lemma 3.15 gives us that \( X \) is a Gaussian process. Defining new processes by

\[ Y(t) = a(t), \quad Z(t) = \frac{X(t)}{a(t)} = \left( x_0 + \int_0^t b(s) ds + \int_0^t c(s) dW(s) \right), \]

we get that \( dY(t) = a'(t) dt \) and \( dZ(t) = b(t) dt + c(t) dW(t) \). The Ito product rule applied to \( X = YZ \) gives that

\[
\begin{align*}
    dX(t) &= Y(t) dZ(t) + Z(t) dY(t) + dY(t) dZ(t) \\
          &= a(t) b(t) dt + a(t) c(t) dW(t) + Z(t) a'(t) dt + 0 \\
          &= \left( a(t) b(t) + \frac{a'(t)}{a(t)} X(t) \right) dt + a(t) c(t) dW(t).
\end{align*}
\]

Looking at

\[ X(t) = (1 - t) \int_0^t \frac{1}{1 - s} dW(s), \tag{1} \]

we see that this fits the general analysis if we put:

\[ x_0 = 0, \quad a(t) = 1 - t \quad \text{(so } a'(t) = -1), \quad b(t) = 0, \quad \text{and } c(t) = \frac{1}{1 - t}. \]

Therefore

\[ dX(t) = \left( 0 + \frac{-1}{1 - t} X(t) \right) dt + (1 - t) \frac{1}{1 - t} dW(t) = \frac{-X(t)}{1 - t} dt + dW(t). \]

By the general results \( X \) is Gaussian, and since for any \( t < 1 \) we have \( \int_0^t (1/(1 - s))^2 ds < \infty \), Björk’s Proposition 3.4 shows that \( \mathbf{E}(X(t)) = 0 \) for all \( t < 1 \).

Now consider \( 0 \leq s \leq t < 1 \) (it is implied, though not entirely clear, that \( s \) and \( t \) are meant to satisfy this) at look at

\[
\text{cov}(X(s), X(t)) = (1 - t)(1 - s) \mathbf{E} \left( \int_0^s \frac{dW(u)}{1 - u} \times \int_0^t \frac{dW(v)}{1 - v} \right) \quad \text{(because of 0-mean)}
\]

\[
= (1 - t)(1 - s) \mathbf{E} \left( \int_0^s \frac{dW(u)}{1 - u} \times \left( \int_0^t \frac{dW(v)}{1 - v} + \int_s^t \frac{dW(v)}{1 - v} \right) \right)
\]

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The last term produces something with mean 0 (condition on $F_s$, use the tower law and Björk’s Proposition 3.7), so it suffices to look at
\[
\mathbb{E} \left( \int_0^s \frac{dW(u)}{1-u} \times \int_0^t \frac{dW(v)}{1-v} \right) = \mathbb{E} \left( \int_0^s \frac{1}{(1-u)^2} du \right) \quad \text{(by the Ito Isometry, Björk (3.13))}
\]
\[
= - \int_1^{1-s} \frac{1}{u^2} du = \int_1^{1-s} \frac{1}{v^2} dv \quad \text{("} v = 1 - u\text{")}
\]
\[
= \left[ -\frac{1}{v} \right]_1^{1-s} = \left( -1 + \frac{1}{1-s} \right) = \frac{s}{1-s}.
\]
When collecting the terms this shows that
\[
\text{cov}(X(s), X(t)) = s(1-t) \quad \text{for} \quad 0 \leq s \leq t < 1.
\]
Notice that $X(t) \xrightarrow{L^2} 0$ for $t \to 1$, so it’s natural to define $X(1)$ to be 0.
If we define $Y$ through
\[
Y(t) = W(t) - tW(1),
\]
then it is clear from the properties of Brownian motion that $Y$ is a 0-mean Gaussian process. If we remember that $\text{cov}(W(u), W(v)) = \min(u, v)$, then for $0 \leq s \leq t \leq 1$ we can also readily find the covariance
\[
\text{cov}(Y(s), Y(t)) = \text{cov}(W(s) - sW(1), W(t) - tW(1))
= \text{cov}(W(s), W(t)) - \text{cov}(W(s), tW(1))
- \text{cov}(sW(1), W(t)) + \text{cov}(sW(1), tW(1))
= s - ts - st + ts = s(1-t).
\]
For the last part (that wasn’t a question, but . . .) note first that
\[
(X(s), X(t), X(1))^\top \sim N_3 \left( 0, \begin{pmatrix} s & s & s \\ s & t & t \\ s & t & 1 \end{pmatrix} \right) \quad \text{for} \quad 0 \leq s \leq t \leq 1.
\]
Using standard results about conditioning in the normal distribution (fx Section 10.39 in Ernsts Stat 1A-noter) we get that $Z^\top = (X(s), X(t)|X(1) = 0$ is two-dimensionally normally distributed with mean 0 and covariance-matrix
\[
\begin{pmatrix} s & s & s \\ s & t & t \end{pmatrix} - \begin{pmatrix} s & t \end{pmatrix} 1^{-1} \begin{pmatrix} s & t \end{pmatrix} = \begin{pmatrix} s & s \\ s & t \end{pmatrix} - \begin{pmatrix} s^2 & st \\ st & t^2 \end{pmatrix} = \begin{pmatrix} s(1-s) & s(1-t) \\ s(1-t) & t(1-t) \end{pmatrix},
\]
and that gives the result.
With a little sleight of hand the techniques in this exercise can be used to write down SDEs for other processes of “bridge type”, i.e. processes whose starting and finishing points are known, e.g. an Ornstein-Uhlenbeck-bridge.