Mathematical Finance, Fall 2001, Homemade Exercise 2.1, Solution

After correcting for typos we have that
\[ X(t) = a(t) \left( x_0 + \int_0^t b(s)ds + \int_0^t c(s)dW(s) \right). \]

Note that Björk’s Lemma 3.15 gives us that \( X \) is a Gaussian process. Defining new processes by
\[ Y(t) = a(t), \quad Z(t) = \frac{X(t)}{a(t)} = \left( x_0 + \int_0^t b(s)ds + \int_0^t c(s)dW(s) \right), \]
we get that \( dY(t) = a'(t)dt \) and \( dZ(t) = b(t)dt + c(t)dW(t) \). The Ito product rule applied to \( X = YZ \) gives that
\[
dX(t) = Y(t)dZ(t) + Z(t)dY(t) + dY(t)dZ(t) - \frac{a'(t)}{a(t)}X(t)dt + \frac{a'(t)c(t)}{a(t)}X(t)dW(t).
\]
Looking at
\[ X(t) = (1 - t)\int_0^t \frac{1}{1-s}dW(s), \tag{1} \]
we see that this fits the general analysis if we put:
\[ x_0 = 0, \quad a(t) = 1 - t \quad \text{(so }a'(t) = -1\text{),} \quad b(t) = 0, \quad \text{and } c(t) = \frac{1}{1-t}. \]

Therefore
\[ dX(t) = \left( 0 + \frac{-1}{1-t}X(t) \right)dt + (1-t)\frac{1}{1-t}dW(t) - \frac{X(t)}{1-t}dt + dW(t). \]

By the general result \( X \) is Gaussian, and since for any \( t < 1 \) we have \( t(1(1-s))^2ds < \infty \), Björk’s Proposition 3.4 shows that \( \mathbb{E}(X(t)) = 0 \) for all \( t < 1 \).

Now consider \( 0 \leq s \leq t < 1 \) (it is implied, though not entirely clear, that \( s \) and \( t \) are meant to satisfy this) at look at
\[
\text{cov}(X(s), X(t)) = (1-t)(1-s)\mathbb{E}\left( \int_0^s dW(u) \int_0^t 1-u \frac{dW(v)}{1-v} \right) \quad \text{(because of 0-mean)}
\]
\[ - (1-t)(1-s)\mathbb{E}\left( \int_0^s dW(u) \int_0^t 1-u \frac{dW(v)}{1-v} \right) \]
\[ \left( \begin{array}{c} s \\ s \end{array} \right) \left( \begin{array}{cc} s & s \\ s & l \\ t & t \end{array} \right) \sim N_2 \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \quad \text{for } 0 \leq s \leq t \leq 1. \]

Using standard results about conditioning in the normal distribution (see Section 10.30 in Eörsényi Stat 1A-tome) we get that \( Z^T = (X(s), X(t))|X(1) = 0 \) is two-dimensionally normally distributed with mean \( 0 \) and covariance matrix
\[
\left( \begin{array}{cc} s & s \\ s & l \\ t & t \end{array} \right) \sim \mathbb{E}\left( \left( \begin{array}{c} s \\ s \end{array} \right) \left( \begin{array}{c} s \\ s \end{array} \right) \right) = \left( \begin{array}{cc} s^2 & st \\ st & t^2 \end{array} \right) = \left( \begin{array}{cc} (1-s) & s(1-t) \\ s(1-t) & t(1-t) \end{array} \right), \]
and that gives the result.

With a little sleight of hand the techniques in this exercise can be used to write down SDEs for other processes of “bridge type”, i.e. processes whose starting and finishing points are known, e.g. an Ornstein-Uhlenbeck bridge.