where \( \mu \) and \( \sigma \) are \textit{functions} and \( W^P \) is a 1-dimensional BM under the “real world” probability measure \( P \). So \( r \) is Markov wrt. its own filtration.

Suppose all kinds of ZCB exist. The “formal” equation \( P(t; T) = \mathbb{E}^Q(\exp(-\int_t^T r(u)du)) \) gives the conjecture that

\[
P(t; T) = F(t, r(t); T)
\]

for some smooth \textit{function} \( F \) (of 3 variables)

Hide \( T \)-dependence in superscript and use Ito to get

\[
\frac{dF^T}{F^T} = \left( \frac{F^T_t + \mu F^T_r + \frac{1}{2} \sigma^2 F^T_{rr}}{F^T} \right) dt + \frac{\sigma F^T_r}{F^T} dW^P(t)
\]

\[\begin{align*}
\frac{dF^T}{F^T} &= \left( \frac{F^T_t + \mu F^T_r + \frac{1}{2} \sigma^2 F^T_{rr}}{F^T} \right) dt + \frac{\sigma F^T_r}{F^T} dW^P(t)
\end{align*}\]

\[\begin{align*}
\text{Björk Chapter 21:} \quad & \text{Short rate models; generalities.} \\
Why is there no “(dr-dynamics) \Rightarrow (df-dynamics)” in Proposition 20.5? (There isn’t one!) \\
Surprising? Not really. Think in terms of \# traded assets \& \# sources of risk. \\
Empirically: Variations in “the” short rate “explains” “a large percentage” of the “variation” of the whole term structure. (“The tail wagging the dog”) \\
Is all lost? Certainly not. We get consistency relation between ZCBs of different maturities. (So nice we may forget we have a “problem” at all.) \\
Look (first) at the case where

\[
dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW^P(t)
\]
(symmetric in $S$), and

$$
\frac{dV}{V} = \frac{\alpha^S \sigma_T - \alpha^T \sigma^S}{\sigma_T - \sigma^S} \, dt.
$$

must $= r(t)$ otherwise arbitrage

Rewriting

$$
\frac{\alpha^S - r(t)}{\sigma^S} = \frac{\alpha^T - r(t)}{\sigma^T}
$$

LHS doesn't depend on $T$, RHS doesn't depend on $S \Rightarrow$ the ratio is independent of maturity:

$$
\frac{\alpha^S - r(t)}{\sigma^S} := \lambda(r(t); t).
$$

Now make a self-financing portfolio with a $T$-ZCB and an $S$-ZCB. $V$ is the value process & $(u_T, u_S)$ relative portfolio weights. From Chapter 6 we have

$$
\frac{dV}{V} = u_T \frac{dF^T}{F^T} + u_S \frac{dF^S}{F^S} = (u_T \alpha^T + u_S \alpha^S)dt + (u_T \sigma^T + u_S \sigma^S)dW^P(t)
$$

By construction we must have $u_T + u_S = 1$, but still 1 degree of freedom. A clever choice is $u_T \sigma^T + u_S \sigma^S = 0$.

The $dW^P$-term vanishes, we get

$$
u_T = \frac{-\sigma^S}{\sigma^T - \sigma^S}
$$
where
\[ dr(s) = (\mu - \lambda \sigma) ds + \sigma dW^Q(s) \]

Note: Clearly \( P(t; T) / \beta(t) \) is a \( Q \)-martingale.

Writing
\[
\frac{dP(t; T)}{P(t; T)} = \alpha^T(t; T) dt + \sigma^T(t, T) dW^P = r(t) dt + \sigma^T(t, T) \left( dW^P + \frac{\alpha^T - r(t)}{\sigma^T} dt \right)
\]
shows that pieces fit.

Remark on multi-dimensional model.

We’re still not very concrete.

\( \lambda \): “market price of risk”; interpretation as excess expected return. Has to be exogenously specified. Usually: Postulate form that gives same structure under \( P \) & \( Q \).

Substitute back and get the term structure PDE:
\[
F^T_t + (\mu - \lambda \sigma) F^T_r + \frac{1}{2} \sigma^2 F^T_{rr} = r F^T \quad \text{and} \quad F^T(T; r) = 1
\]

This may be Feynman-Kac represented (by \[ \text{BE 5.12} \]) & we may change measure:
\[
F(t, r(t); T) = E^Q(\exp(-\int_t^T r(s) ds))
\]