6.5 Pricing the European call

We now have the martingale measure $Q$ in place and hence the value at time $t$ of a European call with maturity $T$ is given in an arbitrage-free model by

$$C_t = \frac{1}{R^{T-t}} E^Q (\max(0, S_T - K) | \mathcal{F}_t).$$

Using this fact we get the following

**Proposition 22** Let the stock and money market account be as described in section 6.4. Then the price of a European call option with exercise price $K$ and maturity date $T$ is given as

$$C_t = \frac{1}{R^{T-t}} \sum_{i=0}^{T-t} \binom{T-t}{i} q^i (1-q)^{T-t-i} \max(0, S_t u^i d^{T-t-i} - K).$$

**Proof.** Since the money market account and $S_0$ are deterministic, we have that we get all information by observing just stockprices, or equivalently the $U$’s, i.e. $\mathcal{F}_t = \sigma(S_1, \ldots, S_t) = \sigma(U_1, \ldots, U_i)$. By using (6.3) twice we can write

$$S_T = S_t u^{(N_T-N_t)} d^{(T-t)-(N_T-N_t)} = S_t u^Z d^{(T-t)-Z},$$

where $Z = N_T - N_t = \sum_{j=t+1}^{T} U_j \sim b(q; (T-t))$, and $Z$ is independent of $\mathcal{F}_t$ (because the $U$’s are independent). Therefore

$$R^{T-t} C_t = E^Q ((S_T - K)^+ | \mathcal{F}_t) = E^Q ((S_t u^Z d^{(T-t)-Z} - K)^+ | \mathcal{F}_t).$$

At this point in the narrative we need something called “the useful rule”. It states the following: Suppose we are given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, a $\sigma$-algebra $\mathcal{F}$, an $\mathcal{F}$-measurable random variable $X$ and a random variable $Y$ that is independent of $\mathcal{F}$. Define the function $g : \mathbb{R} \rightarrow \mathbb{R}$ by $g(x) = E(f(x, Y))$. Then $E(f(X, Y) | \mathcal{F}) = g(X)$. We then use this in the above expression with $S_t$ playing the role of $X$, $Z$ as $Y$, and $f(x, y) = (x u^y d^{(T-t)-y} - K)^+$. By using the general transformation rule for discrete random variables $E(h(Y)) = \sum_{y_i} h(y_i) P(Y = y_i)$, and the fact that $Z$ is $Q$-binomially distributed we get in the notation of “the useful rule” that

$$g(x) = \sum_{i=0}^{T-t} \binom{T-t}{i} q^i (1-q)^{(T-t)-i} (x u^i d^{(T-t)-i} - K)^+, $$

and the desired result follows. □
CHAPTER 6. OPTION PRICING

We rewrite the expression for $C_0$ using some handy notation. Let $a$ be the smallest number of upward jumps needed for the option to finish in the money, i.e.

$$a = \min_{j \in \mathbb{N}} \{ j \mid S_0 u^j d^{T-j} > K \}$$

$$= \min_{j \in \mathbb{N}} \{ j \mid j \ln u + (T - j) \ln d > \ln(K/S_0) \}$$

$$= \min_{j \in \mathbb{N}} \{ j \mid j > \ln(K/(S_0 d^T))/\ln(u/d) \}$$

$$= \left\lceil \ln \left( \frac{K}{S_0 d^T} \right) \right\rceil + 1.$$

Letting

$$\Psi(a; T, q) = \sum_{i=a}^{T} \binom{T}{i} q^i (1 - q)^{T-i},$$

we may write (you may want to check the first term on the RHS)

$$C_0 = S_0 \Psi(a; T, q') - \frac{K}{R^T} \Psi(a; T, q)$$

(6.4)

where

$$q' = \frac{u}{R} q.$$

Using put-call parity gives us the price of the European put:

**Corollary 23** The price of a European put option with $T$ periods to maturity, exercise price $T$ and the stocks as underlying security has a price at time 0 given by

$$P_0 = \frac{K}{R^T} (1 - \Psi(a; T, q)) - S_0 (1 - \Psi(a; T, q'))$$

Note that our option pricing formulae use $T$ to denote the number of periods until maturity. Later, we will be more explicit in relating this to actual calendar time.

### 6.6 Hedging the European call

We have already seen in a two period model how the trading strategy replicating a European call option may be constructed. In this section we simply state the result for the case with $T$ periods and we then note an interesting