Chapter 4

Arbitrage pricing in a one-period model

One of the biggest success stories of financial economics is the Black-Scholes model of option pricing. But even though the formula itself is easy to use, a rigorous presentation of how it comes about requires some fairly sophisticated mathematics. Fortunately, the so-called binomial model of option pricing offers a much simpler framework and gives almost the same level of understanding of the way option pricing works. Furthermore, the binomial model turns out to be very easy to generalize (to so-called multinomial models) and more importantly to use for pricing other derivative securities (i.e. different contract types or different underlying securities) where an extension of the Black-Scholes framework would often turn out to be difficult. The flexibility of binomial models is the main reason why these models are used daily in trading all over the world.

Binomial models are often presented separately for each application. For example, one often sees the "classical" binomial model for pricing options on stocks presented separately from binomial term structure models and pricing of bond options etc.

The aim of this chapter is to present the underlying theory at a level of abstraction which is high enough to understand all binomial/multinomial approaches to the pricing of derivative securities as special cases of one model.

Apart from the obvious savings in allocation of brain RAM that this provides, it is also the goal to provide the reader with a language and framework which will make the transition to continuous-time models in future courses much easier.
4.1 An appetizer.

Before we introduce our model of a financial market with uncertainty formally, we present a little appetizer. Despite its simplicity it contains most of the insights that we are about to get in this chapter.

Consider a one-period model with two states of nature, $\omega_1$ and $\omega_2$. At time $t = 0$ nothing is known about the time state, at time $t = 1$ the state is revealed. State $\omega_1$ occurs with probability $p$. Two securities are traded:

- A stock which costs $S$ at time 0 and is worth $uS$ at time 1 in one state and $dS$ in the other.

- A money market account which costs 1 at time 0 and is worth $R$ at time 1 regardless of the state.

Assume $0 < d < R < u$. (This condition will be explained later.) We summarize the setup with a graph:

$$
\begin{align*}
  &\omega_1 \left( \begin{array}{c} R \\ uS \end{array} \right) \\
  &\omega_2 \left( \begin{array}{c} R \\ dS \end{array} \right) \\
  &\left( \begin{array}{c} 1 \\ S \end{array} \right)
\end{align*}

p

1 - p

Now assume that we introduce into the economy a European call option on the stock with exercise price $K$ and maturity 1. At time 1 the value of this call is equal to

$$
C_1(\omega) = \begin{cases} 
[us - K]^+ & \text{if } \omega = \omega_1 \\
[ds - K]^+ & \text{if } \omega = \omega_2
\end{cases}
$$
4.1. AN APPETIZER.

We will discuss options in more detail later. For now, note that it can be thought of as a contract giving the owner the right but not the obligation to buy the stock at time 1 for $K$.

To simplify notation, let $C_u = C_1(\omega_1)$ and $C_d = C_1(\omega_2)$. The question is: What should the price of this call option be at time 0? A simple portfolio argument will give the answer: Let us try to form a portfolio at time 0 using only the stock and the money market account which gives the same payoff as the call at time 1 regardless of which state occurs. Let $(a, b)$ denote, respectively, the number of stocks and units of the money market account held at time 0. If the payoff at time 1 has to match that of the call, we must have

$$a(uS) + bR = C_u$$
$$a(dS) + bR = C_d$$

Subtracting the second equation from the first we get

$$a(u - d)S = C_u - C_d$$

i.e.

$$a = \frac{C_u - C_d}{S(u - d)}$$

and algebra gives us

$$b = \frac{1}{R} \frac{uC_d - dC_u}{(u - d)}$$

where we have used our assumption that $u > d$. The cost of forming the portfolio $(a, b)$ at time 0 is

$$\frac{(C_u - C_d)}{S(u - d)} S + \frac{1}{R} \frac{uC_d - dC_u}{(u - d)} \cdot 1$$

$$= \frac{R (C_u - C_d)}{R (u - d)} + \frac{1}{R} \frac{uC_d - dC_u}{(u - d)}$$

$$= \frac{1}{R} \left[ \frac{R - d}{u - d} C_u + \frac{u - R}{u - d} C_d \right].$$

We will formulate below exactly how to define the notion of no arbitrage when there is uncertainty, but it should be clear that the argument we have just given shows why the call option must have the price

$$C_0 = \frac{1}{R} \left[ \frac{R - d}{u - d} C_u + \frac{u - R}{u - d} C_d \right]$$
Rewriting this expression we get
\[
C_0 = \left( \frac{R - d}{u - d} \right) \frac{C_u}{R} + \left( \frac{u - R}{u - d} \right) \frac{C_d}{R}
\]
and if we let
\[
q = \frac{R - d}{u - d}
\]
we get
\[
C_0 = q \frac{C_u}{R} + (1 - q) \frac{C_d}{R}.
\]
If the price were lower, one could buy the call and sell the portfolio \((a, b)\), receive cash now as a consequence and have no future obligations except to exercise the call if necessary.

Some interesting features of this example will be much clearer as we go along:

- The probability \(p\) plays no role in the expression for \(C_0\).
- A new set of probabilities
  \[
  q = \frac{R - d}{u - d} \quad \text{and} \quad 1 - q = \frac{u - R}{u - d}
  \]
  emerges (this time we also use that \(d < R < u\)) and with this set of probabilities we may write the value of the call as
  \[
  C_0 = E^q \left[ \frac{C_1(\omega)}{R} \right]
  \]
  i.e. an expected value using \(q\) of the discounted time 1 value of the call.
- If we compute the expected value using \(q\) of the discounted time 1 stock price we find
  \[
  E^q \left[ \frac{S(\omega)}{R} \right] = \left( \frac{R - d}{u - d} \right) \frac{1}{R} (uS) + \left( \frac{u - R}{u - d} \right) \frac{1}{R} (dS) = S
  \]
  The method of pricing the call really did not use the fact that \(C_u\) and \(C_d\) were call-values. Any security with a time 1 value depending on \(\omega_1\) and \(\omega_2\) could have been priced.
4.2 The single period model

The mathematics used when considering a one-period financial market with uncertainty is exactly the same as that used to describe the bond market in a multiperiod model with certainty: Just replace dates by states.

Given two time points \( t = 0 \) and \( t = 1 \) and a finite state space

\[
\Omega = \{ \omega_1, \ldots, \omega_S \}.
\]

Whenever we have a probability measure \( P \) (or \( Q \)) we write \( p_i \) (or \( q_i \)) instead of \( P(\{\omega_i\}) \) (or \( Q(\{\omega_i\}) \)).

A security price system is a vector \( \pi \in \mathbb{R}^N \) and an \( N \times S \) matrix \( D \) where we interpret the \( i \)’th row \((d_{i1}, \ldots, d_{iS})\) of \( D \) as the payoff at time 1 of the \( i \)’th security in states \( 1, \ldots, S \), respectively. The price at time 0 of the \( i \)’th security is \( \pi_i \). A portfolio is a vector \( \theta \in \mathbb{R}^N \) whose coordinates represent the number of securities bought at time 0. The price of the portfolio \( \theta \) bought at time 0 is \( \pi \cdot \theta \).

**Definition 18** An arbitrage in the security price system \((\pi, D)\) is a portfolio \( \theta \) which satisfies either

\[
\pi \cdot \theta \leq 0 \in \mathbb{R} \text{ and } D^\prime \theta > 0 \in \mathbb{R}^S
\]

or

\[
\pi \cdot \theta < 0 \in \mathbb{R} \text{ and } D^\prime \theta \geq 0 \in \mathbb{R}^S.
\]

A security price system \((\pi, D)\) for which no arbitrage exists is called arbitrage-free.

**Remark 1** Our conventions when using inequalities on a vector in \( \mathbb{R}^k \) are the same as described in Chapter (3).

When a market is arbitrage-free we want a vector of prices of ‘elementary securities’ - just as we had a vector of discount factors in Chapter (3).

**Definition 19** \( \psi \in \mathbb{R}^S_{++} \) (i.e. \( \psi \gg 0 \)) is said to be a state-price vector for the system \((\pi, D)\) if it satisfies

\[
\pi = D\psi
\]

Clearly, we have already proved the following in Chapter (3):
Proposition 7 A security price system is arbitrage-free if and only if there exists a state-price vector.

Unlike the model we considered in Chapter (3), the security which pays 1 in every state plays a special role here. If it exists, it allows us to speak of an 'interest rate':

Definition 20 The system $(\pi, D)$ contains a riskless asset if there exists a linear combination of the rows of $D$ which gives us $(1, \ldots, 1) \in \mathbb{R}^S$.

In an arbitrage-free system the price of the riskless asset $d_0$ is called the discount factor and $R_0 = \frac{1}{d_0}$ is the return on the riskless asset. Note that when a riskless asset exists, and the price of obtaining it is $d_0$, we have

$$d_0 = \theta_0^\top \pi = \theta_0^\top D\psi = \psi_1 + \cdots + \psi_S$$

where $\theta_0$ is the portfolio which constructs the riskless asset.

Now define

$$q_i = \frac{\psi_i}{d_0}, i = 1, \ldots, S$$

Clearly, $q_i > 0$ and $\sum_{i=1}^S q_i = 1$, so we may interpret the $q_i$'s as probabilities. We may now rewrite the identity (assuming no arbitrage) $\pi = D\psi$ as follows:

$$\pi = d_0 Dq = \frac{1}{R_0} Dq,$$

where $q = (q_1, \ldots, q_S)^t$.

If we read this coordinate by coordinate it says that

$$\pi_i = \frac{1}{R_0} (q_1 d_{i1} + \cdots + q_S d_{iS})$$

which is the discounted expected value using $q$ of the $i$’th security’s payoff at time 1. Note that since $R_0$ is a constant we may as well say ”expected discounted...”.

We assume throughout the rest of this section that a riskless asset exists.

Definition 21 A security $c = (c_1, \ldots, c_S)$ is redundant given the security price system $(\pi, D)$ if there exists a portfolio $\theta_c$ such that $D^\top \theta_c = c$.  


4.2. THE SINGLE PERIOD MODEL

Proposition 8 Given an arbitrage-free system $\left(\pi, D\right)$ and a redundant security $c$. The augmented system $\left(\hat{\pi}, \hat{D}\right)$ obtained by adding $\pi_c$ to the vector $\pi$ and $c \in \mathbb{R}^S$ as a row of $D$ is arbitrage-free if and only if

$$\pi_c = \frac{1}{R_0} (q_1 c_1 + \ldots \ldots + q_S c_S) \equiv \psi_1 c_1 + \ldots \ldots + \psi_S c_S.$$

Proof.

Assume $\pi_c < \psi_1 c_1 + \ldots \ldots + \psi_S c_S$.

Buy the security $c$ and sell the portfolio $\theta_c$. The price of $\theta_c$ is by assumption higher than $\pi_c$, so we receive a positive cash-flow now. The cash-flow at time 1 is 0. Hence there is an arbitrage opportunity.

If $\pi_c < \psi_1 c_1 + \ldots \ldots + \psi_S c_S$ reverse the strategy. ■

Given a security price system $(\pi, D)$.

Definition 22 The market is complete if for every $y \in \mathbb{R}^S$ there exists a $\theta \in \mathbb{R}^N$ such that

$$D^t \theta = y$$

i.e. if the rows of $D$ (the columns of $D^t$) span $\mathbb{R}^S$.

Proposition 9 If the market is complete and arbitrage-free, there exists precisely one state-price vector $\psi$.

The proof is exactly as in Chapter (3).

We are ready to do contingent claims pricing! Here is how it is done in a one-period model: Construct a set of securities (the $D$-matrix,) and a set of prices. Make sure that $(\pi, D)$ is arbitrage-free. Make sure that either

(a) the model is complete, i.e. there are as many linearly independent securities as there are states

or

(b) the contingent claim we wish to price is redundant given $(\pi, D)$.

Clearly, (a) implies (b) but not vice versa. (a) is almost always what is done in practice. Given a ”contingent claim” $c = (c_1, \ldots, c_S)$. Now compute the price of the contingent claim as

$$\pi(c) = \frac{1}{R_0} E^q (c) \equiv \frac{1}{R_0} \sum_{i=1}^{S} q_i c_i.$$
where \( q_i = \frac{\psi_i}{d_0} \equiv R_0 \psi_i \). The portfolio generating the claim is the solution to \( D' \theta_c = c \), and since we can always in a complete model reduce the matrix to an \( S \times S \) invertible matrix without changing the model this can be done by matrix inversion.

Let us return to our example in the beginning of this chapter: The security price system is

\[
\left\{ \begin{pmatrix} 1 \\ S \end{pmatrix}, \begin{pmatrix} R & R \\ uS & dS \end{pmatrix} \right\}.
\]

For this to be arbitrage-free, proposition (7) tells us that there must be a solution \((\psi_1, \psi_2)\) with \( \psi_1 > 0 \) and \( \psi_2 > 0 \) to the equation

\[
\begin{pmatrix} 1 \\ S \end{pmatrix} = \begin{pmatrix} R & R \\ uS & dS \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.
\]

\( u \neq d \) ensures that the matrix \( \begin{pmatrix} R & R \\ uS & dS \end{pmatrix} \) has full rank. \( u > d \) can be assumed without loss of generality. We find

\[
\psi_1 = \frac{R - d}{R(u - d)}
\]

\[
\psi_2 = \frac{u - R}{R(u - d)}
\]

and note that the solution is strictly positive precisely when \( u > R > d \) (given our assumption that \( u > d > 0 \)).

Clearly the riskfree asset has a return of \( R \), and

\[
q_1 = R \psi_1 = \frac{R - d}{u - d}
\]

\[
q_2 = R \psi_2 = \frac{u - R}{u - d}
\]

are the probabilities defining the measure \( q \) which can be used for pricing. Note that the market is complete, and this explains why we could use the procedure in the previous example to say what the correct price at time 0 of any claim \((c_1, c_2)\) should be.