Topics in Continuous-Time Finance, Spring 2003

Project 2: Numerical Solution of PDEs SUFFICIENT VERSION

Answers must be handed in no later than at the lectures on Wednesday, April 9, 2003. Groups of 3 are OK; groups of 4 or more are not.

General hint:

Consider the parabolic PDE

\[
F_t(x, t) + \mu(x, t) F_x(x, t) + \frac{1}{2} \sigma^2(x, t) F_{xx}(x, t) - r(x, t) F(x, t) = h(x, t) = (AF)(x, t) 
\]

with terminal condition \( F(x, T) = g(x) \).

We want to solve such problems (sometimes referred to as Cauchy problems) numerically by finite difference methods. As in Seydel Chapter 4, suppose the space/time domain has been discretized, and let \( w_{ij} \) denote the finite difference approximation to \( F(x_i, t_j) \). For some \( \theta \in [0; 1] \) we can construct a finite difference scheme by using

\[
\begin{align*}
F(x_i, t_j) &\approx (1 - \theta) w^j_i + \theta w^{j+1}_i \\
F_t(x_i, t_j) &\approx \frac{w^{j+1}_{i} - w^j_i}{\Delta t} \\
F_x(x_i, t_j) &\approx (1 - \theta) \frac{w^{j+1}_{i+1} - w^{j+1}_{i-1}}{2\Delta x} + \theta \frac{w^{j+1}_{i+1} - w^{j+1}_{i-1}}{2\Delta x} \\
F_{xx}(x_i, t_j) &\approx (1 - \theta) \frac{w^{j+1}_{i+1} - 2w^j_i + w^{j+1}_{i-1}}{\Delta x^2} + \theta \frac{w^{j+1}_{i+1} - 2w^{j+1}_i + w^{j+1}_{i-1}}{\Delta x^2} 
\end{align*}
\]

Do a Taylor-expansion or two to convince yourself (and me) that this leads to a consistent scheme meaning that if you apply the difference operators to the true solution of the PDE in order to approximate its various derivatives, then you get reminder terms that tend to 0 as \( \Delta x \) and \( \Delta t \) tend to 0. Or in the words of Seydel: The local truncation errors vanish in the limit. (You are not asked to determine the exact order at which this happens, so you don’t actually need the PDE.)
Show that these approximations lead to the following system of equations (with a clear tridiagonal structure)

\[ a_{i,j}w_{i-1}^j + b_{i,j}w_i^j + c_{i,j}w_{i+1}^j = \alpha_{i,j}w_{i-1}^{j+1} + \beta_{i,j}w_i^{j+1} + \gamma_{i,j}w_{i+1}^{j+1} + \epsilon_{i,j}, \]

for \( i \)'s that correspond to space-discretization points ranging from the second to the next-to-last, and where

\[
\begin{align*}
  a_{i,j} &= \frac{1 - \theta}{2\Delta x} \left( \mu(x_i, t_j) - \frac{\sigma^2(x_i, t_j)}{\Delta x} \right), \\
  b_{i,j} &= \frac{1}{\Delta t} + (1 - \theta) \left( r(x_i, t_j) + \frac{\sigma^2(x_i, t_j)}{(\Delta x)^2} \right), \\
  c_{i,j} &= \frac{1 - \theta}{2\Delta x} \left( -\mu(x_i, t_j) - \frac{\sigma^2(x_i, t_j)}{\Delta x} \right), \\
  \alpha_{i,j} &= -\frac{\theta}{2\Delta x} \left( -\mu(x_i, t_j) - \frac{\sigma^2(x_i, t_j)}{\Delta x} \right), \\
  \beta_{i,j} &= \frac{1}{\Delta t} - \theta \left( r(x_i, t_j) + \frac{\sigma^2(x_i, t_j)}{(\Delta x)^2} \right), \\
  \gamma_{i,j} &= \frac{\theta}{2\Delta x} \left( \mu(x_i, t_j) + \frac{\sigma^2(x_i, t_j)}{\Delta x} \right), \quad \text{and} \\
  \epsilon_{i,j} &= -h(x_i, t_j) \\
\end{align*}
\]

Now look at the Black-Scholes model for a non-dividend-paying stock,

\[ dS(t) = rS(t)dt + \sigma^{bs} S(t)dW^Q(t), \]

and at strike-\( K \), expiry-\( T \) call-option on the stock. Default parameters are shown below.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Numerical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial stock price</td>
<td>( S(0) )</td>
<td>100</td>
</tr>
<tr>
<td>Volatility</td>
<td>( \sigma^{bs} )</td>
<td>0.20</td>
</tr>
<tr>
<td>Interest rate</td>
<td>( r )</td>
<td>0.05</td>
</tr>
<tr>
<td>Strike/exercise price</td>
<td>( K )</td>
<td>105</td>
</tr>
<tr>
<td>Option expiry</td>
<td>( T )</td>
<td>1</td>
</tr>
</tbody>
</table>

How does this fit into the framework; which PDE does the call-price solve? Find the price of a call option by using the finite difference method just analyzed. You have to think about boundary conditions; Seydel’s equation (4.18) is one way. You will also have to think about/experiment with where to actually put the boundaries.

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Investigate the convergence. (Look at your notes from the lectures on February 24, or look at the papers by Ole Østerby that I have linked to on the course homepage.) What is the global order in the space and the time direction; or rather: Is there a clear picture? Try $\theta = 0, 1/2, 1$. What happens if you try to extrapolate?

Often there are good reasons to put $F_{xx} = 0$ on the boundaries. How would you implement that?

There are good reasons why we like to use $\theta = 1/2$, ie. the Crank/Nicolson-scheme. (Seydel’s Theorem 4.4 tells why.) In this case your analysis above should show that the convergence happens in a non-monotone/non-smooth way. One way to try to fix this to start the solution procedure at time $T - \Delta t$ (instead of $T$) and then use

$$g(x) = \text{BSformula}(x, \Delta t),$$

where $\text{BSformula}(x, \Delta t)$ denotes the true Black/Scholes-price for the in question option with $\Delta t$ time to expiry. How does that work?

Another Volatility

Sometimes the B/S-model is extended into the so-called constant elasticity of variance (CEV) model, where to volatility function is

$$\sigma(x, t) = \sigma^{CEV} x^{\gamma}$$

Modify (if necessary) you finite difference code to allow for this form of volatility.

Assume $\gamma = 1/2$ and $\sigma^{CEV} = 2.047$.

Find the implied volatility for put-options with different degrees of “money-ness”, ie. different strike-prices (and other model characteristics as before). Plot implied volatility against strike price. What happens for (deep) out-of-the-money puts? (And why the seemingly strange choice of $\sigma^{CEV}$?)