Put $z_k^{(n)} = \ln S_k^{(n)}$, and consider a specification

$$z_k^{(n)} = z_{k-1}^{(n)} + \mu \Delta_n - \frac{1}{2} h_{k-1}^{(n)} \Delta_n + \sqrt{h_{k-1}^{(n)}} \sqrt{\Delta_n} \epsilon_k^{(n)}$$  \hspace{1cm} (1)

where

$$h_k^{(n)} = \beta_0^{(n)} + \beta_1^{(n)} h_{k-1}^{(n)} + \beta_2^{(n)} (\epsilon_k^{(n)} - c^{(n)})^2$$  \hspace{1cm} (2)

This is a slightly different and extended specification compared to last time. There’s a plethora of GARCH’es. This is called NGARCH; a large negative return raises volatility more than a large positive.

With our knowledge of the Euler-scheme, it’s not hard to conjecture the limit of (1).

\[\text{GARCH can approximate 2-D diffusion.}\]

Literature: Nelson (1990) is standard reference; Duan (1997) has many more examples; this follows Frey (1997).

Set-up: $[0; T]$ is split op into $n$ intervals $[t_{k-1}; t_k]$ of length $\Delta_n = T/n$. Consider a GARCH-model living on these points. What happens when $n \to \infty$?

Problem: No explicit time-step length dependence in our GARCH-definition. We have to introduce that ourselves.

For each $n$ we let $\{\epsilon_k^{(n)}\}$ be a sequence of independent $N(0,1)$-distributed variables.

Now name to coefficients (“left to right”) $a_0^{(n)}, \ldots, a_3^{(n)}$, put $y_k^{(n)} = \frac{((\epsilon_k^{(n)})^2 - 1)}{\sqrt{2}}$ and write

$$h_k^{(n)} - h_{k-1}^{(n)} = \frac{a_0^{(n)}}{\Delta_n} \Delta_n + \frac{a_1^{(n)}}{\Delta_n} h_{k-1}^{(n)} \Delta_n + \frac{a_2^{(n)}}{\sqrt{\Delta_n}} h_{k-1}^{(n)} \sqrt{\Delta_n} \epsilon_k^{(n)}$$  \hspace{1cm} (3)

Note that

$$\text{Var}(y_k^{(n)}) = \frac{E((\epsilon_k^{(n)})^4)}{2} - 1 - 2(\epsilon_k^{(n)})^2 = \frac{3 + 1 - 2}{2} = 1$$

(2) is harder, but the the trick is to write

$$(\epsilon_k^{(n)} - c^{(n)})^2 = ((\epsilon_k^{(n)})^2 - 1) + (-2c^{(n)}) \epsilon_k^{(n)} + (c^{(n)})^2 + 1$$

We have that $E((\epsilon_k^{(n)})^2 - 1)) = E(\epsilon_k^{(n)}) = 0$, and by normality (just symmetry will do, actually) further that

$$E(\epsilon_k^{(n)}((\epsilon_k^{(n)})^2 - 1)) = E((\epsilon_k^{(n)})^3) - E(\epsilon_k^{(n)}) = 0,$$

so we have a decomposition into uncorrelated variables.

Now plug this into equation (2) & subtract $h_{k-1}^{(n)}$ on both sides to get

$$h_k^{(n)} - h_{k-1}^{(n)} = \beta_0^{(n)} + (\beta_1^{(n)} + \beta_2^{(n)}((c^{(n)})^2 + 1) - 1)h_{k-1}^{(n)}$$

$$+ \beta_2^{(n)} h_{k-1}^{(n)} ((\epsilon_k^{(n)})^2 - 1) + (-2c^{(n)}) \epsilon_k^{(n)}$$
Note that (1) and (3) is almost the Euler-scheme for (4)-(5). \( Y_{k}^{(n)} \) has the right mean (0) the right variance (1) and the right covariance with \( \epsilon \) (0). It only it were Gaussian ...

Fortunately, it can be shown that convergence (appropriately uniformly) of local first and second order moments suffices if the jumps of the discrete processes vanish in the limit and here the \( \sqrt{\Delta} \) factor takes care of that. (Standard reference is the somewhat incomprehensible Ethier & Kurz; I suspect results from Kloeden & Platen would do too.)

But in a nutshell: (4)-(5) is the limit.

Assume that as \( \Delta_n \to 0 \) we have

\[
\frac{\alpha_0^{(n)}}{\Delta_n} = \frac{\beta_1^{(n)}}{\Delta_n} = \frac{\alpha_1^{(n)}}{\sqrt{\Delta_n}} = \frac{\alpha_2^{(n)}}{\sqrt{\Delta_n}} = \frac{\alpha_3^{(n)}}{\sqrt{\Delta_n}} = \alpha_3.
\]

Let \( W_1 \) and \( W_2 \) be independent Brownian motions. Look at the solution to the SDE

\[
dZ(t) = (\mu - \frac{1}{2} h_t) dt + \sqrt{\Delta_n} dW_1(t) \quad (4)
\]

\[
dh(t) = (\alpha_0 + \alpha_1 h_t) dt + \alpha_3 h_t dW_1(t) + \sqrt{2 \alpha_2 h_t} dW_2(t) \quad (5)
\]

What do the convergence conditions on the \( \alpha \)'s mean?

The condition \( \frac{\alpha_0^{(n)}}{\Delta_n} \to \alpha_0 \) is OK if

\[
\beta_0^{(n)} = \Delta_n \beta_0.
\]

Unless

\[
\frac{\beta_2^{(n)}}{\Delta_n} = \beta_2 \sqrt{\Delta_n} + o(\sqrt{\Delta_n}) \quad \text{and} \quad c^{(n)} = c \sqrt{\Delta_n} + o(\sqrt{\Delta_n})
\]

then there's no convergence.

If \( \beta_2 = 0 \) and \( c = 0 \) the we get a degenerate limit model; \( h \) isn't stochastic. You can perfectly legally assume this. It's just a question of how rapidly \( \beta_2^{(n)} \) and \( c^{(n)} \) tend to 0.
Estimation example

Crude application where we just detrend and forget the "1/2-from-Ito"-term and set $c = 0$. Daily observations so $\Delta = 1/252$. Recall from (1) that in this case

$$\frac{Z_t - Z_{t-\Delta}}{\sqrt{\Delta}} \sim N(0, h(t))$$

> library(tseries,survival)
> tst<-garch((returnSP500-mean(returnSP500))/sqrt(dt))
> tst$coef
a0 (beta_0'n) a1 (beta_2'n) b1 (beta_2'n)
0.0002212473 0.0684786126 0.9252386066

So: $\sqrt{1/252} \times 0.06847 = \beta_2 \Rightarrow \beta_2 = 1.087 \ (\text{in front of } dW_2)$

And: $(1/252) \times 0.0002212473 \Rightarrow \beta_0 = 0.05575432$

And for $\beta_1$

$$0.92523 = 1 + \beta_1 \Delta_n - \beta_2 \sqrt{\Delta_n}$$

so $\beta_1 = 252 \times (0.92523 + 0.068478612 - 1) = -1.585$

The long term level for $h$ is

$$-\frac{\alpha_0}{\alpha_1} = \frac{\beta_0^{(n)}}{1 - \beta_2^{(n)} - \beta_2^{(n)}} = 0.0351,$$

and its $\sqrt{\alpha}$ is $0.188$. Finally

$$dh(t) = 1.585(0.0351 - h(t))dt + 1.537h(t)dW_2(t)$$