2) if we know the transition density
\[ p(t, y|x, s), \]
ie. \( y \mapsto p(t, y|x, s) \) is the density of \( X(t) \) given \( X(s) = x \)
Ex: Ornstein-Uhlenbeck/Vasicek (easy; Gaussian) Cox-Ingersoll-Ross (hard; non-central-\( \chi^2 \))

For simulation purposes, we’d especially like 1). That rarely happens.

Can we construct approximations, \( Y^n \), such that

- \( Y^n \) is close/converges to \( X \)
- the \( Y^n \)'s can be simulated (note: each \( Y^n \) is a stochastic process)

New question: Converges in what sense? And how fast?

- Strong convergence of order \( \gamma \) if
  \[ E(|X(T) - Y^n(T)|) = O((1/n)^\gamma) \]
- Weak convergence of order \( \gamma \) wrt. a function \( g \)
  \[ |E(g(X(T)) - g(Y^n(T)))| = O((1/n)^\gamma) \]

Result: The Euler-scheme converges strongly with order 1/2 and weakly (wrt. all polynomials) with order 1.

Strong schemes have some path-wise convergence properties. Enough said.

---

**Simulation of Diffusion Processes** (Seydel Ch. 3)

Diffusion: Sol’n of SDE
\[ dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW, \quad X(0) = x_0 \]
where \( \mu \) and \( \sigma \) are functions (of appropriate dimensions; but let’s say everything is 1-d).

Sol’n is a stochastic process. When can we reasonably say we know the solution?

1) if we know a deterministic function \( f \) such that
\[ \forall t : X(t) = f(W(t), t). \]
Geo. BM: \( f(x, t) = x_0 \exp((\mu - \sigma^2/2)t + \sigma x) \)

Assume \( \mu \) and \( \sigma \) not (calendar) time-dependent, fix some \( T \), an \( n \), put \( \Delta_n = T/n \) and \( t^n_i = i\Delta_n \).

Natural idea: \( dW \)'s are approximately iid \( N(0, dt) \). This leads to the Euler scheme
\[ Y^n(t^n_i) = Y^n(t^n_{i-1}) + \mu(Y^n(t^n_{i-1}))\Delta_n + \sigma(Y^n(t^n_{i-1}))\Delta W(t_i) \]
where (of course) \( Y^n(0) = x_0 \) and \( \Delta W(t_i) = W(t_i) - W(t_{i-1}) \) (so the \( \Delta W \)'s are iid \( N(0, \Delta_n) \)).

Notational nightmare when written in full, but easy to simulate. Good news: The Euler scheme converges as \( n \to \infty \). (Closely connected to how SDE sol’n’s are defined.)
Simulations for Geo. BM:

- Fairly clear that there’s both weak and strong convergence.
- Evident that the strong order is 1/2.
- Less evident that the weak order, but with a little suggestive graphics ...

What’s with the different orders? Different forms of convergence are different, so ...
What’s with the “half-order”? “Non-integer order” is something we’re used to for BM; “$dW \sim dt^{1/2}$”.

Now do the same for SDEs: Ito-Taylor-expansion.

Ito’s formula says that for any $C^2$-function $f$ we have

$$f(X(t)) = f(x_0) + \int_0^t (\mathbb{L}^0 f)(X(s)) \, ds + \int_0^t (\mathbb{L}^1 f)(X(s)) \, dW(s),$$

where the operators $\mathbb{L}^0$ and $\mathbb{L}^1$ are defined through

$$(\mathbb{L}^0 f)(x) = \mu(x) f'(x) + \frac{1}{2} \sigma^2(x) f''(x)$$

and $(\mathbb{L}^1 f)(x) = \sigma(x) f'(x)$.

Using this for $f(x) = x$ gives the almost tautology

$$X(t) = x_0 + \int_0^t \mu(X(s)) \, ds + \int_0^t \sigma(X(s)) \, dW(s),$$

Can we improve the orders? Yes. But we have to study closely the terms we have thrown away.

Thrown away? Recall a Taylor-expansion:

$$f(t_0 + \Delta t) = f(t_0) + f'(t_0) \Delta t + f''(t_0) \frac{(\Delta t)^2}{2} + \cdots + f^{(n)}(t_0) \frac{(\Delta t)^n}{n!} + O((\Delta t)^n)$$

How to “prove” this: Fundamental theorem of calculus over & over.

Dif’eqn-link: Sometimes we can easily express RHS in terms of the coefficients. Ex: If $f' = a(t)$ then $f(t_0 + \Delta t) \approx f(t_0) + a(t_0) \Delta t + a'(t_0)(\Delta t)^2/2$. 
But this can be calculated explicitly ("classical exercise")

$$\int_0^t W(s)dW(s) = \frac{1}{2}(W(t)^2 - t)$$

This suggests we use the (or: a) Milstein scheme:

$$Y^n(t^n_i) = Y^n(t^n_{i-1}) + \mu(Y^n(t^n_{i-1}))\Delta t + \sigma(Y^n(t^n_{i-1}))\Delta W(t_i)$$
$$+ \frac{1}{2}\sigma(Y^n(t^n_{i-1}))\sigma'(Y^n(t^n_{i-1}))(\Delta W(t_i)^2 - \Delta t)$$

Result: The Milstein scheme has weak order 1 (wrt. polynomials) and strong order 1.

But now use the result for $f = \mu, \sigma$ inside the integrals:

$$X(t) = x_0 + \int_0^t (\mu(x_0) + \int_0^s (\mathbb{L}\mu)(X(u))du + \int_0^s (\mathbb{L}^1\mu)(X(u))dW(u))ds$$
$$+ \int_0^t (\sigma(x_0) + \int_0^s (\mathbb{L}\sigma)(X(u))du + \int_0^s (\mathbb{L}^1\sigma)(X(u))dW(u))dW(s)$$

where (for instance) $(\mathbb{L}^1\sigma) = \sigma\sigma'$. Collecting all the double integrals in a remainder $R$, and then throwing that away, we get the Euler scheme.

We can apply the Ito-trick again to see that the $R$-term contains among other things

$$\sigma(x_0)\sigma'(x_0) \int_0^t \int_0^s dW(u)dW(s) = \sigma(x_0)\sigma'(x_0) \int_0^t W(s)dW(s)$$

Clear strategy: Apply Ito to the 3 remaining double integrals in the remainder. If we do that we have to look at terms like

$$\int_0^t \int_0^s du ds = \frac{1}{2}t^2$$
$$\int_0^t \int_0^s dW(u)ds = \int_0^t W(s)ds := Z(t) \sim N(0, \frac{t^3}{3})$$ (small exercise)
$$\int_0^t \int_0^s dW(u)dW(s) = \int_0^t sdW(s) = W(t) - Z(t)$$ (Ito right to left)
Can show that \((W(t), Z(t))\) is 2-d Gaussian and that \(\text{cov}(W(t), Z(t)) = \frac{1}{2} t^2\). In simulations \((t \rightarrow \Delta t)\) this can be achieved by using

\[
\Delta W = \sqrt{\Delta t} \epsilon_1 \quad \text{and} \quad \Delta Z = \frac{1}{2} \Delta t^{3/2} (\epsilon_1 + \frac{1}{\sqrt{3}} \epsilon_2),
\]

where \(\epsilon\)'s “independent standard normal”.

In this way you get Seydel’s (3.13). This scheme has weak order 2 and (still only) strong order 1.

(But if you add \(\frac{1}{2} b(b'' + (b')^2) \left( \frac{1}{3} \Delta W^2 - \Delta \right) \Delta W \) it gets strong order 1.5.)

- You can do this ’till the cows come home. Is there a pattern (like for regular Taylor-expansion)? Not a simple one: The operators \(\mathbb{L}^0\) and \(\mathbb{L}^1\) don’t commute and we get \(n\)-fold recursive BM-integrals.

- In theory you only have to “hard-code” a particular scheme once. Then you just specify \(\mu\) and \(\sigma\) functions and some of their derivatives. Of course in practice ...

- Higher dimensions: Everything but the Euler-scheme gets very hard or messy.

- Variance reduction along the path in a scheme does not make sense. But using “a \(W\)-path and it’s negative” is OK; the same is simulation a known SDE-solution based on the \(W\)-path.

- Computational efficiency trade-off in MC-pricing: Duffie & Glynn shows that for a weak order \(\gamma\) scheme it is computationally optimal to have

\[
\#\text{paths} \propto \# \text{ discretization steps}^{2\gamma}
\]

Moral: For (say) an order-2 scheme, the discretization error is not our main concern.

- Prefer strong schemes if something must be calculated along the path.

- Despite all our calculations we have at best given an indication a local truncation error order. Real convergence (order) proofs are left to the experts (Kloeden & Platen, Talay).