**Topics in Continuous-Time Finance, Spring 2003**

**Project 1: Effects of Discrete Hedging**

Answers must be handed in no later than at the lectures on Monday, February 24, 2003. Groups of 3 are OK; groups of 4 or more are not.

Throughout we consider the Black-Scholes model for a non-dividend-paying stock,

\[ dS = \mu S dt + \sigma S dW. \]

We assume a deterministic (continuously compounded) interest rate of \( r \), and look at a strike-\( K \), expiry-\( T \) call-option on the stock. Default parameters are shown below.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Numerical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial stock price</td>
<td>( S(0) )</td>
<td>100</td>
</tr>
<tr>
<td>Stock price (proportional) drift</td>
<td>( \mu )</td>
<td>0.05</td>
</tr>
<tr>
<td>Volatility</td>
<td>( \sigma )</td>
<td>0.20</td>
</tr>
<tr>
<td>Interest rate</td>
<td>( r )</td>
<td>0.05</td>
</tr>
<tr>
<td>Strike/exercise price</td>
<td>( K )</td>
<td>105</td>
</tr>
<tr>
<td>Option maturity</td>
<td>( T )</td>
<td>1</td>
</tr>
</tbody>
</table>

Comment on “\( \mu = r \)”.

Recall that in order to replicate the call-option in the Black-Scholes model we have to hold

\[ \Delta(t) = \frac{\partial C}{\partial S} = \Phi \left( \frac{\ln(S(t)/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right), \]

units of stock, and adjust our (stock,bank account)-portfolio (continuously) in a self-financing way. Of course, continuous adjustment is hard in practice, but how bad are we off if we do things discretely? To answer this, consider the following algorithm/procedure/plan:
- Chop the time interval between now (time 0) and expiry of the call-option (time T) into N pieces; denote the discretization points $t_i$.

- Simulate stock-price paths, denote the $j$’th by $(S^j(t_i))^N_0$ (i.e. values at the $t_i$’s).

- Suppose that at time 0 somebody gives you an amount of money exactly equal to the Black-Scholes call-price. You use this to buy $\Phi(d_1(S(0), 0))$ units of the stock (whatever extra money you need, you borrow in the bank).

- At time $t_1$ you adjust your portfolio such that you now hold $\Phi(d_1(S^j(t_1), t_1))$ units of the stock. You do this in such a way that extra funds needed (+/-) are borrowed at the bank. (Recall that money in the bank draws interest.)

- Do that all the way up to $t_N = T$, where you liquidate the portfolio. Keep track of the value of the portfolio, say $V^j(t_i)$, along each path. Compare $V^j(t_i)$ to the true Black-Scholes call-price; call the difference $\epsilon^j_N(t_i)$; it may also be referred to as (running) hedge error, the profit/loss or simply “P/L”. Let $\bar{\epsilon}_N^j(t_i) = e^{-rt_i} \epsilon^j_N(t_i)$ denote the discounted hedge error.

Run the simulation for $M = 1000$ paths and make a scatter-plot of the call-option payoff against the terminal portfolio value. Do this for (at least) hedge frequencies $N = 12, 52, 250$. What do you see?

What is $\mathbb{E}^Q(\bar{\epsilon}_N(t_N))$? What is $\mathbb{E}^Q_t(\bar{\epsilon}_N^j(t_m))$ for any $t_l < t_m$? Hint: The $Q$-expected discounted value of a self-financing trading strategy (which?) is a $\mathbb{E}^Q_e$? So what can you (and J. L. W. V. Jensen) say about std.dev.(\bar{\epsilon}_N^j(t)) as a function of $t$?

It is possible (but hard) to show that as a function of $N$, std.dev.(\bar{\epsilon}_N^j(t_N)) behaves asymptotically as $1/\sqrt{N}$. Illustrate that numerically.

What happens if $\mu \neq r$? Illustrate & explain.

What happens if we hedge with a wrong volatility? Experiment & explain.

Do the discrete delta-hedging exercise again, but this time for an option whose pay-off is $1_{S(T) \geq K}$, i.e. a digital option. Now how does the standard deviation of the terminal hedge error behave as a function the hedging frequency? So how would you feel about hedging digitals?