Alan Lewis

ALGORITHMS

Asian Connections

Welcome to a series of articles on topics in mathematical finance, mostly about *algorithms* for valuing derivative securities. I'm interested in models and computational methods: how can we realistically value the options that the marketplace is interested in? Your feedback is important to me – if you have comments, corrections, or perhaps some topics that you would like to see discussed, please email me at the address at the end of the article.

In this article, I take up the topic of Asian options, which are options on averages. This class of option is popular in the OTC market, and has a number of flavors. It's also very popular with the mathematical finance community – I think the appeal of it to the theorists is that it's a tricky problem which is just on the edge of solvability by analytic methods. In fact, an exact solution has recently been found by Vadim Linetsky, who also has a comprehensive bibliography. In this article, I show an alternative "quick and dirty" approach to his exact solution.

I discuss a basic case where we have a European-style put or call option with a fixed strike price K. On expiration, the call option buyer receives the amount, if any, that the average security price over the option's lifetime exceeds the strike. Of course, the security could be anything – oil prices or some other commodity prices or the S&P500 index are all good examples. One appeal of these options, compared to plain vanilla puts and calls, is that they are better suited to more thinly traded markets or where price manipulation on expiration is a concern.

How does the S&P500 fit into the "thin market" scenario? Well, it doesn't, but nevertheless Asians on the S&P500 have become a popular basis for the equity-linked annuities of insurance companies. The insurance companies use them to offer their customers some equity market 'participation' with no losses (unless the insurance company defaults). The annuity issuer then turns around and hedges this obligation at a lower cost than a plain vanilla option.

Several years ago, I published an article in Mathematical Finance that solved a couple of problems that Robert Merton had posed, but only partially solved, back in the 70's. One of these was the problem of the option on a stock which paid out its dividends continuously, at a constant dollar rate, say D dollars per year. This innocent sounding problem was posed in the standard Black-Scholes setting of constant volatility σ and that's our setting here. In this model, the stock price follows the stochastic differential equation (SDE) $dS = (rS - D)\,dt + \sigma S\,dB(t)$, for $S > 0$. Here r is an interest rate and $dB(t)$ is a Brownian motion process. If the price reaches $S = 0$, the dividend is dropped and the process stops (i.e., we have *absorption*).

Over relatively short time frames, this payout model is a rough approximation to the payout on broad-based indexes. That's because there are lots of little dividends if you "own" the index and companies are slow to change their dividend policies. For an individual company, you might object that such a dividend policy couldn't be realistic because, if the company gets into trouble and the stock price gets very low, the board will drop or alter the dividend. Fair enough, but consider the recent case of WorldCom's MCI tracking stock, which maintained a \$2.40 per year dividend (\$0.60 quarterly) as the stock price fell dramatically, all the way down until the stock price reached the \$2.00 range.

Then they had to drop it. Anyway, (1) hey, it's only a model, and (2) it helps solve the more interesting Asian option problem.

Now during the review process for the paper [Lewis, 1998], one of the anonymous referees noted that this problem had some connections to the Asian option problem, but didn't give details. I told myself this subject would be worth pursuing at some point, but didn't do much with it. Over the years, other researchers have noted this same connection. For example, Alexander Lipton mentions it in a Risk magazine article [1999] and Daniel Dufresne has mentioned it in conversation. Then, most recently, I took on a consulting assignment involving Asian options – time to really look at this Asian connection!

The first connection with the Asian option problem is that both problem are intimately concerned with the behavior of the SDE $dX = (\alpha X - \delta)dt + \sigma X dB(t)$, where $\delta > 0$. We have already shown this SDE for the continuous dividend problem. Let's now see how it arises for the Asian option problem.

For simplicity, we'll also use S as the security price underlying the Asian option. But we'll tackle only the simplest case where this security pays no dividends. Then, S evolves under its risk-neutral measure as $dS = rS\,dt + \sigma S\,dW(t)$, where $W(t)$ is another standard Brownian motion. The Euro-style Asian call value is *freshly* written at time $t = 0$ and expires at $t = T$. At $t > 0$, the call is *seasoned*. Whether fresh or seasoned, the call value is given formally by the usual discounted expectation

$$C_t = e^{-r(T-t)}\mathbb{E}_t\left[\left(\int_0^T S_u \frac{du}{T} - K\right)^+\right], \qquad 0 \le t \le T. \tag{1}$$

The notation is $x^+ = \max(x, 0)$, K is the strike price, and $\mathbb{E}_t[\ldots]$ is a time-t expectation. Notice that the payoff depends upon the continuous average of the stock price over the holding period. The goal is to derive an exact computational formula for (1).

The PDE of Rogers and Shi

Consider some time $t > 0$ in (1). Since the option is seasoned, one already knows part of the integral in (1), namely the part from 0 to t. So it's natural to break it up into two pieces. This was the beginning of the argument of Rogers and Shi [1995], who made use of the dimensionless variable $X_t = (K - \bar{S}_t)/S_t$, where $\bar{S}_t \equiv \int_0^t S_u du/T$ is the *average-to-date* (my notation). A useful interpretation of X_t is that it measures the (negative of the) moneyness in terms of *shares* – in other words, one has simply changed numeraire from dollars to shares[2]. At expiration, if $X_T < 0$, the option is in-the-money and $-X_T$ is the payoff in shares. Note that if S_t were very small then X_t would be arbitrarily large and positive. On the other hand $\int_0^t S_u du/T$ could be arbitrarily large, which would drive X_t to minus infinity. Hence X_t ranges over $-\infty < X < \infty$.

Introduce the martingale M_t defined by

$$M_t = \mathbb{E}_t\left[\left(\int_0^T S_u \frac{du}{T} - K\right)^+\right] = S_t\,\phi(X_t, t),$$

$$\text{where } \phi(x, t) = \mathbb{E}_t\left[\left(\int_t^T \frac{S_u}{S_t}\frac{du}{T} - x\right)^+\right]. \tag{2}$$

Now Ito's formula yields $dX = [-1/T - (r - \sigma^2)X]\,dt - \sigma X\,dW(t)$. Apply-ing Ito again to M_t, and using the fact that M_t is a martingale (and hence the dt coefficient of its Ito expansion vanishes), Rogers and Shi derived the PDE:

$$0 = \phi_t + r\phi + \frac{1}{2}\sigma^2 x^2 \phi_{xx} - (rx + 1/T)\,\phi_x, \qquad -\infty < x < \infty, \tag{3}$$

where the subscripts indicate differentiation. It's convenient to remove a term, so defining $f(x, t) = \exp[-r(T-t)]\,\phi(x, t)$, we also have

$$0 = f_t + \frac{1}{2}\sigma^2 x^2 f_{xx} - (rx + 1/T) f_x, \qquad -\infty < x < \infty. \tag{4}$$

To summarize the transformations, we have the Asian call value given by

$$C_t(S_t, K) = e^{-r(T-t)} S_t\,\phi(X_t, t) = S_t f(X_t, t)\,. \tag{5}$$

As with X_t, the function $f(X_t, t)$ has a change-of-numeraire interpretation. From (5), you can see that it simply gives the call option value in terms of shares. Then, the value in terms of, say, dollars is obtained by multiplying by the exchange rate in dollars per share.

What are the PDE boundary conditions? First, at expiration, $X_T = [K - \int_0^T S_u du/T]/S_T$. If $X_T > 0$, then the stock price average ends up out-of-the-money. The call option has expired worthless, so $f(x, T) = 0$. On the other hand, if $X_T < 0$, then $C_T(S_T, K) = -S_T X_T$, so $f(X_T, T) = -X_T$. To summarize, we have the terminal condition $f(x, T) = x^- = -\min[x, 0]$. The boundary points $x = \pm\infty$ are singular points and no other boundary conditions are required! But, see below how we actually use our boundary condition.

Comparing the Two Problems

Eqn. (4) is a very simple looking PDE. In finance, when we see a PDE, we immediately think of its solution as an expectation over a stochastic process defined by an SDE. Since no other boundary conditions are required, this solution and SDE are given by

$$f(X_0, T) = \mathbb{E}_0\,[f(X_T, T)], \qquad \text{where } f(x, T) = x^-, \tag{6}$$

$$dX = -\left(rX + \frac{1}{T}\right) dt + \sigma X dB(t), \tag{7}$$

and $\mathbb{E}[...]$ is an expectation with a measure under which $dB(t)$ is a Brownian motion. Note that this SDE has *changed* from the dX_t evolution given below eqn (2) – recall: $dX = -[(r-\sigma^2)X + 1/T]\,dt - \sigma X\,dW(t)$. That's because the change of numeraire from dollars to shares is associated with a change of measure. Under this change of measure, the Brownian motion process changes, too: $dW_t \to d\tilde{W}_t + \sigma\,dt$, which cancels a drift term. Then, we took the liberty of writing $dB_t = -d\tilde{W}_t$. At this stage, we have the following relationship between our two problems:

Problem:	SDE: $dX = (\alpha X - \delta)\,dt + \sigma X dB(t)$
(I) Stock with constant dollar payout rate.	Parameters: $\alpha = r > 0$, $\delta = D > 0$, $\sigma > 0$. Domain: $X > 0$.
(II) Asian option	Parameters: $\alpha = -r < 0$, $\delta = 1/T > 0$, $\sigma > 0$. Domain: $-\infty < X < \infty$.

So, the two associated SDEs have the same functional form. However, the multiplicative drift parameters α have opposite signs, and the domains are different. As it turns out, the sign of α plays no role in my previous results Lewis [1998] that we need here. In addition, I next show how to reduce the Asian option problem to one using *only* information from the half-line $X > 0$.

Reduction of the Asian option problem to a half-line (X > 0)

With a plain vanilla European-style option, you never know until expiration if you will end up in-the-money. Asian options are different. With an Asian option, once the average-to-date $\bar{S}_t$ exceeds the strike, then it will certainly exceed the strike on expiration since its non-decreasing in time. But $\bar{S}_t > K$ implies $X_t < 0$. This tells us that the sample path behavior of X_t is very interesting. It starts out at $x_0 > 0$ and has some positive probability of hitting $x = 0$. If it does hit $x = 0$, then it leaves the region $x > 0$ forever, and stays negative until expiration.

When $x < 0$, we have the simple and well-known computation:

$$\begin{aligned}\phi(x,t) &= \mathbb{E}_t\left[\left(\int_t^T \frac{S_u}{S_t}\frac{du}{T} - x\right)^+\right] = E_t\left[\left(\int_t^T \frac{S_u}{S_t}\frac{du}{T} - x\right)\right] \\ &= \mathbb{E}_t\left[\left(\int_t^T \exp\left[(r - \frac{1}{2}\sigma^2)(u-t) + \sigma W_{u-t}\right]\frac{du}{T} - x\right)\right] \\ &= \frac{e^{r(T-t)} - 1}{rT} - x.\end{aligned}$$

In particular, this gives us the simple "boundary" condition, if only $x = 0$ were a boundary:

$$f(x=0,t) = \frac{1 - e^{-r(T-t)}}{rT}. \tag{8}$$

To use this condition, we need to make $x = 0$ a boundary. From (6), we see that $f_t = f(x_t, t)$ is a martingale. A nice property of martingales is that they remain martingales if the expectation is taken not just at a fixed time, but an arbitrary *stopping time* τ. We choose $\tau = \min[T, t_0]$, where t_0 is the moment when the process $dX = -(rX + 1/T)\,dt + \sigma X\,dB(t)$ first reaches the origin, starting from x_0. Using (8) and the fact that $f(x,T) = 0$, for $x \geq 0$, we have

$$f(x_0,T) = \mathbb{E}_0\,[f(x_\tau,\tau)] = \int_0^T p(x_0,t)\,\frac{1 - e^{-r(T-t)}}{rT}\,dt, \tag{9}$$

where $p(x_0,t)$ is the hitting time density to hit the origin in the interval $(t, t+dt)$ starting from x_0. This hitting time density has a cumulative distribution $A(x,t) = \Pr\{x_{t_0} = 0,\ t_0 \leq t\}$, where $p(x,t) = \partial A(x,t)/\partial t$ and "Pr" denotes "probability". With a parts integration, we rewrite (9) as

$$\begin{aligned}f(x,T) &= \int_0^T \left(\frac{\partial}{\partial t}A(x,t)\right)\frac{1 - e^{-r(T-t)}}{rT}\,dt \\ &= \frac{1}{T}\int_0^T A(x,t)\,e^{-r(T-t)}\,dt, \qquad x > 0.\end{aligned} \tag{10}$$

From the SDE correspondences discussed above, we know that the hitting time distribution $A(x,t)$ is identical to the absorption probability (i.e., bankruptcy probability) for a stock that pays a continuous dollar dividend at the rate 1/T, with multiplicative drift parameter of -r . Eqn (10) reduces the Asian option problem to this hitting time problem on the positive half-line. But I solved this hitting time problem in Lewis [1998]. Next, I show how to use this solution to obtain the exact Asian option solution of Linetsky's in a relatively painless manner.

The Hitting Time Distribution

In this section, I merely quote the previously obtained results from [1998], using a little better notation. We need notation for the hitting time distribution that displays the parameters. So, consider again the process $dX = (\alpha X - \delta)\,dt + \sigma X\,dB_t$, starting from $X(0) = x > 0$, where α, δ, and σ are constant parameters and dB_t is a Brownian motion. We require that $(\delta, \sigma) > 0$, while α may have any sign. Now, let $A_{\alpha,\delta}(x,t)$ be the cumulative distribution for $X(t)$ to hit the origin for the first time prior to t. (The parametric dependence upon σ need not be displayed). We also define the Laplace transform

$$\hat{A}_{\alpha,\delta}(x,s) = \mathfrak{L}[A_{\alpha,\delta}] = \int_0^\infty \exp(-sT)\,A_{\alpha,\delta}(x,T)\,dT.$$

Then, here are the two results we need[3]:

Proposition 1. The Laplace transform $\hat{A}_{\alpha,\delta}(x,s)$ is given by

$$\hat{A}_{\alpha,\delta}(x,s) = \frac{\Gamma(c_s - a_s)}{\Gamma(c_s)} \frac{1}{s} \left(\frac{\gamma}{x}\right)^{a_s} M\left(a_s, c_s, -\frac{\gamma}{x}\right), \tag{11}$$

using

$$\beta = \frac{2\alpha}{\sigma^2}, \quad \gamma = \frac{2\delta}{\sigma^2}, \quad a_s = \frac{\beta - 1}{2} + \frac{1}{2}\sqrt{(\beta-1)^2 + \frac{8s}{\sigma^2}},$$

$$c_s = 1 + \sqrt{(\beta-1)^2 + \frac{8s}{\sigma^2}}. \tag{12}$$

How to perform the Laplace transform inversion is discussed below. Right now, let's just give the result:

Proposition 2. The hitting distribution $A_{\alpha,\delta}(x,t)$ is given by

$$A_{\alpha,\delta}(x,t) = 1 - \frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(c_\mu - a_\mu)}{\Gamma(2i\mu)}\right|^2 \times \eta\left(\frac{\gamma}{x}, \mu\right) \frac{\exp\left\{-\left[\frac{\mu^2}{2} + \frac{(\beta-1)^2}{8}\right]\sigma^2 t\right\}}{\mu^2 + \frac{(\beta-1)^2}{4}} \, d\mu, \tag{13}$$

using

$$\beta = \frac{2\alpha}{\sigma^2}, \quad \gamma = \frac{2\delta}{\sigma^2}, \quad a_\mu = \frac{\beta-1}{2} + i\mu, \quad c_\mu = 1 + 2i\mu, \tag{14}$$

and

$$\eta(x,\mu) = e^{-x} x^{a_\mu} U(c_\mu - a_\mu, c_\mu, x). \tag{15}$$

The Call Option Formula as a Transform

Notice that (10) is a convolution integral. That's convenient because the Laplace transform of a convolution is simply the product of the transforms of the two factors. By applying this idea to (10), and using Proposition 1, we have a formula for the freshly written Asian call option as a Laplace inversion:

$$C_0(S_0, K, T) = \frac{S_0}{T} Q\left(r, T, \frac{K}{S_0}\right), \tag{16}$$

where

$$Q(r,T,x) \equiv \int_0^T A_{-r,1/T}(x,t)\, e^{-r(T-t)} dt = \mathfrak{L}^{-1}\left(\frac{\hat{A}_{-r,1/T}(x,s)}{(s+r)}\right)$$
$$= \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{sT} \frac{\Gamma(c_s - a_s)}{\Gamma(c_s)} \frac{1}{s(s+r)} \left(\frac{\gamma}{x}\right)^{a_s} M\left(a_s, c_s, -\frac{\gamma}{x}\right) ds. \tag{17}$$

In (17), $\varepsilon > 0$ is any positive real number to the right of any singularities of the integrand. Also, note that the integrand uses the formulas for a_s and c_s from (12) where $\beta = -2r/\sigma^2$ since we are dealing with the Asian problem.

Similar results were obtained by Geman and Yor [1993] and Lipton [1999] by different routes. However, these authors did not perform the Laplace inversion analytically. In the next section, we show how to do that. Before doing so, it's important to understand why the analytic inversion is worth the trouble. Why not just be content to do the last integral in (17) numerically? In my experience (using Mathematica), the numerical integration of eqn. (17) is well-behaved for intermediate times to expiration, but problematic at very small and very large times. On the other hand, once you do the analytic inversion, you get an expression (still containing an integration) which is problematic only at small times. So, the analytic inversion formula has fewer problems and is, indeed, worth doing.

The Call Option Formula after the Inversion

Our general method is to use the Residue Theorem from complex analysis, with the integration contour shown in Figure 1. This theorem and the word "residue" are explained briefly below. This contour appears quite frequently in a variety of problems in mathematical finance.

We discover this contour by examining the singularities of the integrand in the second line of (17). The singularities are the points where the integrand is no longer analytic. In this case the singularity structure is actually pretty simple. First, there are two simple poles at $s = 0$ and $s = -r$ which you can see from the integrand. Finally, the two parameters a_s and c_s, which are defined in (12) both have square root pieces. This means that we have a branch cut singularity where the square roots vanish, which is at $s = -(\beta-1)^2\sigma^2/8 = -(1 + 2r/\sigma^2)^2\sigma^2/8$. Since the

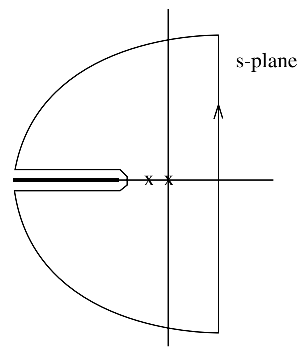

Fig. 1 Inversion Contour for the Asian Option Problem

square root is two-valued, we have to place a branch cut in the s-plane to keep it well-defined; that's the bold line in the figure extending off to $s = -\infty$ along the real axis. The two small x's are the simple poles. So the contour is chosen (i) to enclose the poles and (ii) to extend just above and just below the branch cut. That's it!

By construction, the integrand in (17) is an analytic function everywhere inside the contour in Figure 1 except for the two simple poles. Now the residue theorem from complex analysis states that the integral along such a closed contour, when done in the anti-clockwise sense of Fig. 1, is given by $2\pi i$ times the sum of the residues at the enclosed poles. But the integral is also given by the integrals along the various contour segments. First, there is the straight vertical segment: that's the Laplace inversion $Q(r, T, x)$ from (17) that we are trying to compute. Then, there are the two quarter circles: these integrals vanish as the circles move out to infinity. Finally, there is the sum of the two integrals along the branch cut; let's call their sum the branch cut integral. Putting it all together, we have:

$$\begin{aligned} Q(r, T, x) &= \text{Residue}|_{s=0} + \text{Residue}|_{s=-r} \\ &\quad - \text{Branch cut integral} = Q_1 + Q_2 - Q_3 \end{aligned} \tag{18}$$

Note that the factors of $2\pi i$ have cancelled because we define "Residue" at s_0 to mean the coefficient of $1/(s - s_0)$ of the expressions to the right of the integral sign in (17). Let's consider these three terms in turn.

(i) At $s = 0$, since $\beta = -2r/\sigma^2 < 0$, we have $a_0 = (\beta - 1)/2 + |\beta - 1|/2 = 0$. Similarly, $c_0 = 2 - \beta$, and so the residue of the integrand is $M(0, c_0, -\gamma/x)/r = 1/r$. Hence $Q_1 = 1/r$.

(ii) At $s = -r$, we have $a_{-r} = (\beta - 1)/2 + |\beta + 1|/2$. There are two cases to consider, namely (ii)(a) $r > \sigma^2/2$, which means $\beta < -1$ or (ii)(b) $0 < r < \sigma^2/2$, which means $-1 < \beta < 0$. The result is that (ii)(a) $a_{-r} = -1$, $c_{-r} = -\beta$; and (ii)(b) $a_{-r} = \beta$, $c_{-r} = 2 + \beta$. The resulting expression can be simplified a little, using $M(-1, b, z) = 1 - z/b$, yielding:

$$Q_2 = -\frac{e^{-rT}}{r} \times \begin{cases} 1 - \beta\left(\dfrac{x}{\gamma}\right) & \text{(a) } r > \dfrac{\sigma^2}{2} \\ \dfrac{1}{\Gamma(2+\beta)}\left(\dfrac{\gamma}{x}\right)^{\beta} M\left(\beta, 2+\beta, -\dfrac{\gamma}{x}\right) & \text{(b) } 0 < r < \dfrac{\sigma^2}{2} \end{cases}$$

(iii) Finally, we consider the branch cut integral. This term can be developed using my method in [1998], but instead we will get the answer through a "trick". The trick is to notice that the first equality in (17) gives $Q(r, T, x)$ as an integral over $A_{-r,1/T}(x, t)$ and we already have an expression for $A_{-r,1/T}(x, t)$ at (13). If it was legal to exchange the integration orders, which is dubious, then we would have

$$\begin{aligned} Q(r, T, x) \text{ " = " } &\frac{(1 - e^{-rT})}{r} \\ &- \frac{1}{2\pi}\int_0^{\infty}\int_0^{T}\left|\frac{\Gamma(c_\mu - a_\mu)}{\Gamma(2i\mu)}\right|^2 \eta\left(\frac{\gamma}{x}, \mu\right) \\ &\times \frac{\exp\left\{-\left[\frac{\mu^2}{2} + \frac{(\beta-1)^2}{8}\right]\sigma^2 t - r(T-t)\right\}}{\mu^2 + \frac{(\beta-1)^2}{4}} dt\, d\mu. \end{aligned}$$

Now the dt integration above is trivial, just being a simple exponential. The result is two terms, one from $t = T$ and the other from $t = 0$. The $t = 0$ term leaves a highly improper $d\mu$ integral, with an overall factor of $\exp(-rT)$, which is why we said the exchange of integration order was dubious. But this "improper" term belongs to Q_2 and we already have a good expression for that; so we ignore the improper term. The remaining $t = T$ term is convergent and is, in fact, our branch cut integral:

$$\begin{aligned} Q_3(r, T, x) = &\frac{1}{\pi\sigma^2}\int_0^{\infty}\left|\frac{\Gamma(c_\mu - a_\mu)}{\Gamma(2i\mu)}\right|^2 \\ &\times \eta\left(\frac{\gamma}{x}, \mu\right)\frac{\exp\left\{-\left[\frac{\mu^2}{2} + \frac{(\beta-1)^2}{8}\right]\sigma^2 T\right\}}{\left[\mu^2 + \frac{(\beta-1)^2}{4}\right]\left[\mu^2 + \frac{(\beta+1)^2}{4}\right]} d\mu. \end{aligned} \tag{19}$$

where recall that $\beta = -2r/\sigma^2$, $\gamma = 2/(\sigma^2 T)$. Again, this is just a shorthand way of obtaining the result. The " = " sign really means that both sides have the same Q_3 part. To carefully obtain Q_3, set up the branch cut integrals as I do in [1998].

The expression (19) can be simplified. First note that $c_\mu - a_\mu = 3/2 - \beta/2 + i\mu$. The Gamma function satisfies the basic property $\Gamma(z) = (z-1)\,\Gamma(z-1)$, which produces:

$$\begin{aligned} |\Gamma(c_\mu - a_\mu)|^2 = &\left[\mu^2 + \frac{(\beta-1)^2}{4}\right]\left[\mu^2 + \frac{(\beta+1)^2}{4}\right] \\ &\times \left|\Gamma\left(-\frac{1}{2}(\beta+1) + i\mu\right)\right|^2. \end{aligned}$$

Also, it's known that $|\Gamma(2i\mu)|^{-2} = [2\mu \sinh(2\pi\mu)]/\pi$. Finally, one can remove an overall factor of $\exp(-rT)$, which converts the $(\beta - 1)^2$ term in the argument of the exponential into $(\beta + 1)^2$. The result is

$$C_0(S_0, K, T) = \frac{S_0}{rT}\{1 - e^{-rT}g - e^{-rT}h\},$$

where

$$g = \begin{cases} 1 - \beta\left(\frac{x}{\gamma}\right) & \text{(a) } r > \frac{\sigma^2}{2} \\ \frac{1}{\Gamma(2+\beta)}\left(\frac{\gamma}{x}\right)^{\beta} M\left(\beta, 2+\beta, -\frac{\gamma}{x}\right) & \text{(b) } 0 < r < \frac{\sigma^2}{2} \end{cases}$$

and

$$h = \frac{\beta}{\pi^2} \int_0^\infty \mu \sinh(2\pi\mu)\, \left|\Gamma\left(-\frac{1}{2}(\beta+1)+i\mu\right)\right|^2 \times \eta\left(\frac{y}{x},\mu\right) \exp\left\{-\left[\frac{\mu^2}{2}+\frac{(\beta+1)^2}{8}\right]\sigma^2 T\right\}\, d\mu.$$

Instead of β, Linetsky and Geman and Yor use the parameter $\nu = (2r/\sigma^2) - 1 = -(\beta+1)$. Also, Linetsky's integration variable is $p = 2\mu$. Finally, we note that the variable x always appears in the combination $y \equiv \gamma/x$. Adopting those notations, we have our final result:

Proposition 3. Let $\nu = (2r/\sigma^2) - 1$ and $y = 2S_0/(\sigma^2 TK)$. Then, the value of a freshly written Asian call option (on a non-dividend paying stock) is given by

$$C_0(S_0, K, T) = \frac{S_0}{rT}\{1 - e^{-rT} g + e^{-rT} h\}. \qquad (20)$$

where

$$g = \begin{cases} 1 + \left(\frac{1+\nu}{y}\right) & \text{(a) } \nu > 0 \\ \frac{1}{\Gamma(1-\nu)} \left(\frac{1}{y}\right)^{1+\nu} M(-1-\nu, 1-\nu, -y) & \text{(b) } -1 < \nu < 0 \end{cases}$$

and

$$h = \frac{(1+\nu)}{4\pi^2} \int_0^\infty p \sinh(\pi p)\, \left|\Gamma\left(\frac{1}{2}(\nu + ip)\right)\right|^2 \xi(y,p) \times \exp\left\{-\frac{1}{8}(p^2+\nu^2)\,\sigma^2 T\right\}\, dp,$$

using

$$\xi(y,p) = e^{-y} y^{a_p}\, U(c_p - a_p, c_p, y),\; a_p = -1 - \frac{\nu}{2} + i\frac{p}{2},\; \text{and } c_p = 1 + ip.$$

Proposition 3 only handles the case when the stock is non-dividend paying. Our same method can treat the more general case where the stock pays a constant yield q. Instead of (17), you start with the transform found in Lipton [1999]. For some formulas, see Linetsky.[4]

Computations

The formula (20) was evaluated in Mathematica for two cases. In both cases, we took $S_0 = K = 2$ in order to have some overlap with Linetsky's results. In Case I, we took $r = 0.05$, $\sigma = 0.5$, so that $r < \sigma^2/2$. (All units are annual). In Case II, we took $r = 0.20$, $\sigma = 0.5$, so that $r > \sigma^2/2$. The numerical integrations were done at machine precision with the default accuracy goal. The results for various times to expiration T are shown in Table I, along with some relative running times. We also show the excellent lower bound approximation of Giles Thompson [1998].

The same numerical examples were also re-computed using Linetsky's integral formula [2001] and found to agree. In fact, the formula (20) and the one in [2001] look somewhat different, but they are almost certainly equivalent for the non-dividend paying stock. By varying the upper integration bounds, I believe the Table results are insensitive to that and accurate to all the digits shown.

With a plain vanilla call on a non-dividend paying stock, call prices increase with the time to expiration. In contrast, with the Asian call, you can see a maximum value at intermediate times. That's because, asymptotically, as T gets large, $C_0 \approx S_0/(rT) \to 0$, which may be seen in (20).

When you numerically integrate the exact formula, at least in Mathematica, you find significantly increased evaluation times as $\sigma^2 T$ becomes small. (See the Run Time column in the Table). This is primarily due to the weakening effect of the exponential cutoff term, which

TABLE I. ASIAN CALL OPTION VALUES: EXACT AND LOWER BOUND APPROXIMATIONS

		Case I ($r < \sigma^2/2$)			Case II ($r > \sigma^2/2$)		
Years to Expiration	Run Time	Exact (Eqn 20)	Lower Bound	L. Bound % Error	Exact (Eqn 20)	Lower Bound	L. Bound % Error
100	1	0.391771	–	–	0.100000	–	–
20	1.3	0.790483	0.786401	−0.52	0.457664	0.457524	−0.03
10	1.3	0.694923	0.692542	−0.34	0.622945	0.622508	−0.07
2	2.4	0.350095	0.349779	−0.09	0.430616	0.430391	−0.05
1	6.4	0.246416	0.246298	−0.05	0.299968	0.299869	−0.03
0.5	16	0.172269	0.172226	−0.02	0.203184	0.203145	−0.02
0.25	62	0.120335	0.120320	−0.01	0.137038	0.137024	−0.01
0.10	659	0.075067	0.075064	−0.004	0.082117	0.082113	−0.005
0.01	–	–	0.007308	–	–	0.024013	–
0.001	–	–	0.002306	–	–	0.007383	–

requires that the remaining terms (special functions with complex arguments) be evaluated at more extreme arguments. The result is that the formula becomes increasingly less useful at shorter times to expiration, and this explains the absence of results in the Table at the smallest times. Presumably, other numerical environments would have similar problems with the integral in (20). However, as one sees from the Table, this problem is easily remedied by employing the lower bound approximation, which becomes increasingly *more* accurate at smaller times. (The Run Time for the Lower Bound Approximation is <1 on the same scale.) So, a robust computational scheme can be established which uses the exact formula for $\sigma^2 T$ above some cutoff and a lower bound approximation for $\sigma^2 T$ below.

Naturally one would want to go on and understand the effect of stochastic volatility and jumps on these basic results. Perhaps later ...

END NOTES & REFERENCES

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2. The introduction of this dimensionless variable reduces the dimensionality of the PDE by one. This *similarity* reduction was applied to the average strike Asian by Ingersoll [1987] (see Ch. 17). For more on this topic, see Ch. 11 in Wilmott, et al.

3. We use the following special functions: the Gamma function $\Gamma(z)$, and the hypergeometric functions $M(a, c, x)$ and $U(a, c, x)$. *Also* $|\Gamma(z)|$ *denotes the absolute value or modulus of* $\Gamma(z)$.

4. See Linetsky's eqn. (10) in his preprint dated Nov. 10, 2001. However, I believe there is a typo in that equation in the term containing $1_{\{\nu<-2\}}$. That term should be multiplied by an additional factor of 1/2. This does not effect any of his numerical results.

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