\[ f(t) = (t-t)e^{t-t/2} \]

\[ f_t = -\frac{(x-t)}{2}e^{t-t/2} - e^{x-t/2} = \frac{-t}{2} - e^{t-t/2} \]

\[ f_x = e^{t-t/2} + (x-t) e^{t-t/2} = e^{x-t/2} + \frac{t}{2} \]

\[ f_{xx} = 2e^{t-t/2} + \frac{t}{2} \]

Thus by FPE we have \[ f(t) = f(e, W(t)) \]

\[ dX = \left( f_t + \frac{1}{2} f_{xx} \right) dt + \frac{f_x}{2} dW \]

\[ (W(t) - t + \frac{1}{2})e^{W(t) - t/2} \]

\[ -\frac{t}{2} - e^{t-t/2} + e^{x-t/2} - f/2 = 0 \]

No drift term here.
EXERCISE 3: OPTION PRICING WITH A TWIST

To show replication we must show that $V(T) = g(S(T))$. $(h_1, h_2) \cdot (S, B) = S(t)F_x(t, S(t)) + B(t)(F(t, S(t)) - S(t)F_x(t, S(t)))$, $B = 1$, and the statement follows from the terminal condition. The self-financing condition is

$$dV = (h_1, h_2) \cdot (dS, dB)$$

As $dB \equiv 0$, the RHS is simply $F_x(t, S(t))\sigma(t)S(t)dW^Q(t)$. Since we can calculate the LHS with the Ito formula

$$dV(t) = F_xdS + Ft dt + \frac{1}{2}F_{xx}(dS)^2$$

$$= F_x(t, S(t))\sigma(t)S(t)dW^Q(t) + \frac{1}{2}S^2(t)F_{xx}(t, S(t), t)$$

where the $F$-PDE is used to substitute out $F_t$. Hence the RH and self-financing holds precisely when

$$\frac{1}{2}S^2(t)F_{xx}(t, S(t))(\sigma^2(t) - \bar{\sigma}^2) = 0.$$

In base-case Black/Scholes (with 0 interest rates) and $\bar{\sigma} = \sigma$ is the standard $\Delta$-hedge, and in this case "all the ends meet", i.e., financing and replicates.
EXERCISE 3: OPTION PRICING

2a [10%]
The short & perfectly acceptable answer is to use the 1999-exam questions (with change of sign) to get $\pi(t)$. The slightly longer version goes like this:

$$
\pi(t) = e^{-r(T-t)}E^Q_t \left( S(T) - \frac{1}{T} \int_0^T S(u)du \right)
$$

$$
= S(t) - \frac{e^{-r(T-t)}}{T} \int_0^t S(u)du - \frac{e^{-r(T-t)}}{T} \int_0^T E^Q_t(S(u))du
$$

$$
= S(t) - \frac{e^{-r(T-t)}}{T} \int_0^t S(u)du - \frac{e^{-r(T-t)}}{T} \int_t^T e^{r(u-t)} E^Q_t(e^{-r(u-t)}S(u))du
$$

$$
= S(t) - \frac{e^{-r(T-t)}}{T} \int_0^t S(u)du - \frac{S(t)e^{-rT}}{T} \int_t^T e^{ru}du
$$

$$
= S(t) - \frac{e^{-r(T-t)}}{T} \int_0^t S(u)du - \frac{S(t)(1 - e^{-r(T-t)})}{rT}
$$

Putting $Z(t) = \int_0^t S(u)du$, we have that

$$
\pi(t) = f(t, S(t), Z(t)),
$$

where $f(t, x, z) = x(1 - (1 - e^{-r(T-t)})/(rT)) + e^{-r(T-t)}z/T$. From Björk's Proposition 7.6 we get that a replicating strategy has

$$
f_x(t, S(t), Z(t)) = 1 - \frac{1 - e^{-r(T-t)}}{rT}
$$

units of stock at time $t$ and is kept self-financing by appropriate trading in the bank-account. A priori simple $\Delta$-hedging might appear not to work because the price depends also on past stock-prices. But by Proposition 7.6 it does, the reason being that the "extra stochastic factor" is of a particularly simple form.

**NOTICE**: This calculation does not explicitly use the Black/Sholes model - use it for one of the exercises but for 7.6 you should just calculate $\pi(t)$ (and construct $r$).
EXERCISE 4

Recall the Black-Scholes calculation

\[ \text{IF } dX = \text{DRIFT} \cdot X \, dt + \text{VOL} \cdot X \, dW \]

\[ \text{THEN } \mathbb{E}_t \left( (X(T) - K)^+ \right) \]

\[ = \mathbb{E}_t \left( \text{exp} \left( \frac{\ln \left( \frac{X(t)}{K} \right) + (\text{DRIFT} + \text{VOL}^2/2) \cdot (T-t)}{\text{VOL} \cdot \sqrt{T-t}} \right) \cdot (X(T) - K)^+ \right) \]

\[ - \mathbb{E}_t \left( \text{exp} \left( \frac{\ln \left( \frac{X(t)}{K} \right) + (\text{DRIFT} - \text{VOL}^2/2) \cdot (T-t)}{\text{VOL} \cdot \sqrt{T-t}} \right) \cdot (X(T) - K)^+ \right) \]

\[ \text{NOTE THAT} \]

(a) By ITO rule, \( d \left( \frac{1}{\sigma_2} \right) = \left( -1 + \sigma_2^2 \right) \frac{1}{\sigma_2} \, dt - \sigma_2 \frac{1}{\sigma_2} \, dW_2 \)

(b) By ITO product rule

\[ d \left( \frac{S_1}{S_2} \right) = \frac{S_1}{S_2} \cdot d\left( \frac{1}{\sigma_2} \right) + \frac{1}{S_2} \cdot dS_1 - \frac{1}{S_2} \cdot dS_2 = 0 \]

\[ = \frac{S_1}{S_2} \left[ \sigma_2^2 \, dt + \sigma_2 \, dW_1 - \sigma_2 \, dW_2 \right] \]

\[ = \frac{S_1}{S_2} \left[ \sigma_2^2 \, dt + \sqrt{\text{VOL}_1^2 + \text{VOL}_2^2} \left( \frac{dW_1}{\sqrt{\text{VOL}_1^2}} - \frac{dW_2}{\sqrt{\text{VOL}_2^2}} \right) \right] \]

1-dim B.M.

(Known from Exercise 83)

Hence we can apply with

\[ \text{DISC} \sim r, \ \text{DRIFT} \sim \text{VOL}_2^2, \ \text{VOL} \sim \sqrt{\text{VOL}_1^2 + \text{VOL}_2^2} \]