European option prices are a good “sanity check” when analysing bonds with exotic embedded options.

It’s an old exam question.

Arbitrage-free economy where ZCB prices are driven 1-D BM, i.e.

\[ dP(t, T) = r(t)P(t, T)dt + \sigma_P(t, T)P(t, T)dW^Q(t). \]

Look at a coupon bond with payments \( \alpha_1, \ldots, \alpha_N \) occurring at dates \( T_1, \ldots, T_N \). The price of this coupon bond is

\[ \pi^B(t) = \sum_{i|T_i > t} \alpha_i P(t, T_i). \]

Options on coupon-bearing bonds

Non-trivial extension of ZCB options (from Björk Ch. 19). Uses change of numeraire. Idea developed independently by Jamshidian (JoF, ’89) and Geman (unpublished, same time).

Important because

- Real-life bonds have coupons.
- Swaptions are a special case.

Problem: Even with deterministic ZCB-price volatility, \( \pi^B \) isn’t log-normal (\( \sum \) lognormals \( \neq \) lognormal), so it seems we won’t get a Black-Scholes type-expression (\( Q_t \’s \) are \( \Phi \) at a suitable point.)

“It ain’t necessarily so.”

Assume ZCB-volatility is of the form

\[ \sigma_P(t, T) = (g(T) - g(t))h(t). \]

for some deterministic functions \( g \) (that is increasing & differentiable) and \( h \) (that is positive).

Last ingredient: A strike-\( K \), expiry-\( T \) European call-option on the coupon bond. Let \( \pi^C(t) \) denote its price, (and as usual \( \beta \) be the bank-account). Then

\[
\pi^C(t) = \beta(t)E^Q_t \left( \frac{(\pi^B(T) - K)1_{\pi^B(T)>K}}{\beta(T)} \right) \\
= \sum_{i|T_i > T} \alpha_i \beta(t)E^Q_t \left( \frac{P(T, T_i)1_{\pi^B(T)>K}}{\beta(T)} \right) - \beta(t)KE^Q_t \left( \frac{1_{\pi^B(T)>K}}{\beta(T)} \right) \\
= \sum_{i|T_i > T} \alpha_i P(t, T_i)E^Q_t \left( \frac{P(T, T_i)1_{\pi^B(T)>K}}{P(T, T_i)} \right) - P(t, T)KE^Q_t \left( 1_{\pi^B(T)>K} \right) \\
= \sum_{i|T_i > T} \alpha_i P(t, T_i)Q^T_t (\pi^B(T) > K) - KP(t, T)Q^T_t (\pi^B(T) > K). \tag{1}
\]
Look at the term $Q_t^T(\pi^B(T) > K)$ from (1). For each $T_i$ define the process $Z(\cdot, T, T_i)$

$$Z(t, T, T_i) = \frac{P(t, T_i)}{P(t, T)}$$

By direct application of Theorem 19.8 from Björk we have that

$$dZ(t, T, T_i) = (g(T_i) - g(T)) h(t) Z(t, T, T_i) dW^T(t),$$

where $W^T$ is a BM under the $T$-forward measure.

Using this on the $Z$-process we get:

$$P(T, T_i) = \frac{P(t, T_i)}{P(t, T)} e^{-\frac{1}{2} \int_T^T h^2(u) du + \int_T^t h(u) dW^T(u)}$$

The first term in the “exp” is deterministic.

The stochastic part of the second term, is the same for all $T_i$ and putting $H(t, T) = \int_t^T h^2(u) du$ we may write it as

$$\sqrt{H(t, T)} X,$$

where $X$ is a $N(0,1)$ under $Q^T$ and independent of $\mathcal{F}_t$.

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**Note**

- from Proposition 15.5 in Björk the forward rate volatility is then

$$\sigma_f(t, T) = -\frac{\partial}{\partial T} \sigma^P(t, T) = -g'(T) h(t),$$

i.e. it’s deterministic & multiplicatively separable.

- this volatility form includes (basically: *is*) the Hull/White (extended Vasicek) model.

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Remember that if a process $Y$ solves the stochastic differential equation

$$dY(t) = \mu(t; \omega) Y(t) dt + \sigma(t; \omega) Y(t) dW(t),$$

then for $s \leq t$ we have

$$Y(t) = Y(s) \exp \left( \int_s^t (\mu(u; \omega) - \frac{1}{2} \sigma^2(u; \omega)) du + \int_s^t \sigma(u; \omega) dW(u) \right).$$

(Proof: Ito on “what’s inside the exp-function”.)

Remember that is $\sigma$ is deterministic then

$$\int_t^T \sigma(u) dW(u) \sim N(0, \int_t^T \sigma^2(u) du),$$

and independent of $\mathcal{F}_t$. 
All in all we have found that the last term in (1) is

\[ KP(t, T)\Phi(d(t, T)). \]

How do we find \( d(t, T) \)? We have

\[ -d(t, T) = \pi^{-B}(K) \Leftrightarrow \pi^B(\ldots, -d(t, T)) = \pi^B(\ldots, \pi^{-B}(K)) = K, \]

so we must find the solution to

\[ \pi^B(\ldots, -d^*) = K. \]

But \( \pi^B \) is a function we know explicitly, that’s easy to do numerically (bisecting or “goal seek”’ing in Excel).

\[ \pi^B(T) = \pi^B(T; x)_{x=X} \]

\[ = \sum_{i[T_i > T]} \text{“a pos. fct”} (t, T, T_i) \times e^{(g(T_i) - g(T))\sqrt{H(t, T) \times x}} \bigg|_{x=X}. \]

Note that \( g \) is increasing and the sum is over \( i \)’s such that \( T_i > T \) so \( g(T_i) - g(T) > 0 \) and the \( x \mapsto \pi^B(T; x) \) is a monotonically increasing (with \( \mathbb{R}_+ \) as domain). So it has an inverse function, formally \( \pi^{-B} \), and this function increasing, too.

We then have

\[ Q^T_i(\pi^B(T; X) > K) = Q^T_i(X > \pi^{-B}(K)) = \Phi(-\pi^{-B}(K)). \]

Remarks

The technique will work for other 1-factor models than the Gaussian one (eg. CIR), except you don’t get \( \Phi’s \) but a more complicated distribution function.

The technique will not work (without possibly crude approximations) in a multidimensional setting. Why? Well, first

\[ ax + by = K \]

defines a line, not a single point. Could try rewriting

\[ \sqrt{a^2 + b^2} \left( \frac{a}{\sqrt{a^2 + b^2} x} + \frac{b}{\sqrt{a^2 + b^2} y} \right) = K. \]

Repeated use of the relation

\[ dW^r(t) = dW^Q(t) - \sigma_P(t, \tau)dt. \]

allows us to perform the same analysis on the \( Q^T_i \)-terms in (1) (this is DIY) and arrive at:

| Result | The time-\( t \) price of a call-option on a coupon bearing bond is given by

\[ \pi^C(t) = \sum_{i[T_i > T]} \alpha_i P(t, T_i)\Phi(d^*(t, T_i)) - KP(t, T)\Phi(d^*(t, T)), \]

where for any \( \tau \in \{T, T_1, \ldots, T_n\} \), \( d^*(t, \tau) \) is defined implicitly as the solution to the equation

\[ \sum_{i[T_i > T]} \frac{\alpha_i}{P(t, T_i)} e^{H(t, T)(g(T_i) - g(T))^2 \left( \frac{g(T_i) - g(T)}{g(T_i) - g(T)} - 1 \right) e^{(g(T_i) - g(T))H(t, T)d^*(t, \tau)}} = K. \]
Simple forward rates; LIBOR

A simple forward rate \( L(t; S, T) \) specifies the cash-flow for a loan agreement where

- The agreement is made at time \( t \)
- At time \( S \) the borrower receives $1 (or Euro, or DKK, or ...)
- At time \( T \) the borrower pays back \( 1 + (T - S)L(t; S, T) \)

But then \( z \) would be different for different \( T_i \)'s. Then what: You can do various “rank 1”-approximations. Claus Munk has done nice work.

**PhD-course participants**: How’s that for continuity!

**“Topics ...”-course participants**: Final project: Implement the formula. Then you could:

- Read the original Jamshidian article.
- Compare theoretical prices of European options to observed Bermudan prices (from the embedded options in “realer”).
- See what Claus does. Replicate his numbers.

With \( T = S + \delta \) we may write

\[ L_\delta(t; S), \]

\( L_\delta(t; t) \) is called \((\delta-)\) spot LIBOR.

**Immediate (Technical) Observation**

\( L_\delta(t; T) \) is a \( Q^{T+\delta} \) martingale.

Note that this rate is quoted on a discretely compounded basis. If \( L(0; 1, 1.25) = 0.04 \) then you have to pay back 1.01; if the 0.04 were taken as continuously compounded you’d have to pay back \( \exp(0.25 \times 0.04) = 1.010050 \).

The usual simple no-arbitrage argument (DIY) shows that

\[ 1 + (T - S)L(t; S, T) = \frac{P(t, S)}{P(t, T)} \Rightarrow L(t; S, T) = \frac{1}{T - S} \left( \frac{P(t, S)}{P(t, T)} - 1 \right). \]

Such simple rates are called LIBOR.
Floating rate bonds & Swaps

Look at a tenor structure; a set of dates where something interesting happens

\[ t \quad T_0 \quad T_1 = T_0 + \delta \quad T_i = T_{i-1} + \delta \quad T_N = T_0 + N\delta \]

A floating rate bullet bond has the cash-flows

\[ \delta L_\delta(T_{i-1}; T_i) \]

at \( T_i \) for \( i \leq N-1 \), and \( 1 + L_\delta(T_{N-1}; T_N) \) at date \( T_N \).

The cash-flows are stochastic so finding the arbitrage-free price seems to require a dynamic model.

It doesn’t.

We have

\[ c_i = \frac{1}{P(T_{i-1}, T_i)} - 1 \text{ for } i \leq N-1 \]

and the time-\( t \) value of the “-1” is of course \(-P(t, T_i)\). Now consider the following trading strategy:

- time \( t \): Buy \( 1 \) \( T_{i-1} \)-ZCB (price: \( P(t, T_{i-1}) \))
- time \( T_{i-1} \): Invest the \$1 received in \( T_{i} \)-ZCB. You’ll get \( 1/P(T_{i-1}, T_i) \) units & a net-cash-flow of 0.

Note that this result is easily extended to any type of floating rate bond (e.g., serial or annuity) with deterministic instalment plan.

If \( H(T_i) \) denotes remaining principal and \( A(T_i) \) is the principal repaid at time \( T_i \) then

\[ H(T_{i-1}) = \sum_{j=i}^N A(T_j). \]

The \( T_i \)-cash-flow from the bond is

\[ c_i = A(T_i) + \delta L_\delta(T_{i-1}; T_i) H(T_{i-1}) = A(T_i) + \delta L_\delta(T_{i-1}; T_i) \sum_{j=i}^N A(T_j). \]

A portfolio with \( A(T_i) \) units of the \( T_i \)-bullet has exactly the same cash-flows, and its price (assuming \( t = T_0 \)) is \( \sum_i A(T_i) = H(0) \). So the new bond has par value too.

- time \( T_i \): Sit back and receive \$1 \( 1/P(T_{i-1}, T_i) \) from the \( T_{i} \)-ZCB.
A plain vanilla interest rate swap is a contract that consists of

- A long position in a floating rate bullet (or however many $M$ you want as notional principal)
- A short position in a fixed rate bullet (say with fixed rate $\kappa$).

You can think of this as a contract that swaps floating rate interest payments for fixed rate payments (or vice versa).

The value of the swap contract is

$$V_{\text{swap}}(t) = P(t; T_0) - P(t; T_N) - \sum_{i=1}^{N} \delta \kappa P(t; T_i)$$

In practice this equation is used backwards (at the time of initiation of the swap) to set the fixed rate such that $V_{\text{swap}}(t) = 0$, i.e.

$$\kappa^*(t) = \frac{P(t; T_0) - P(t; T_N)}{\sum_{i=1}^{N} \delta P(t; T_i)}.$$

This is called the (par) swap rate. Note that it is specific to the swap considered; you get different swap rates if you move $T_0$, $\delta$ or $N$ around.

The message is then:

- floating rate bonds trade at par
- swaps can be valued without a dynamic model (there’s no volatility dependence)

A couple of disclaimers/warnings:

It is very important for the “volatility independence” that you swap the exact right rate at the exact right time. Swapping the 6M LIBOR every 3rd month induces volatility dependence. So does moving payments to when they are first known. So-called convexity adjustment try to remedy that.

Swaps can be made a lot more exotic with all kinds of embedded option features & strange floating rates.

Famous disaster: Proctor and Gamble vs. (literally, later) Bankers Trust. (Arguably, the problem here was not really the complexity, but the fact that P&G took a huge gamble on rates staying low.)
If $\gamma$ is 1D & constant (in 1st argument) then $v^2 = \gamma^2(T_{i-1})T_{i-1}$ and the formula is the so-called Black’s formula.

Other assumption: $\gamma(t, T) = \tilde{\gamma}(T - t)$ where $\tilde{\gamma}$ is piecewise constant.

Not clear what a reasonable volatility specification is.

A cap contract is a series of caplets; its price is simply the sum of caplet prices.

For many years, market practice was to price – or at least quote – caps with the Black formula. Here is a formal, arbitrage-free models that supports this.

So one way to to specify an arbitrage-free model is as

$$d\delta(t; T_{i-1}) = \gamma^\top(t; T_{i-1}) L_\delta(t; T_{i-1}) dW^T_i(t)$$

for some deterministic (possible vector-valued) function $\gamma$.

This is called the (lognormal) LIBOR market model.

Put

$$v^2(t, T) = \int_t^T \|\gamma(u; T)\|^2 du.$$ 

Then a standard B/S-like calculation (DIY) shows that

$$\pi_{\text{caplet}}(t; T_{i-1}, \delta, \kappa) = \delta P(t, T_i) \left( L_\delta(t; T_{i-1}) \Phi(d_+) - \kappa \Phi(d_-) \right),$$

where $d_\pm = (\ln(L_\delta(t; T_{i-1})/\kappa) \pm \frac{1}{2} v^2(t, T_{i-1}))/v(t, T_{i-1})$.

The models are actually more complicated than they look:

- Strange bond price dynamics

$$\sigma_P(t, T) = -\sum_{k=1}^{\lfloor (T-t)/\delta \rfloor} \frac{\delta L_\delta(t, T - \delta k)}{1 + \delta L_\delta(t, T - \delta k)} \gamma(t, T - \delta k).$$

Not a Markovian structure. So simulation requires a lot of book-keeping.

- Lognormality is not preserved on measure changes. And if 3M LIBOR has lognormal volatility structure, then 6M LIBOR hasn’t.

Papers with the model by [Miltsersen, Sandmann, Sondermann], [Brace, Gatarek, Musiela] and [Jamshidian] appeared virtually simultaneously in 1997.

Immediate hit. Understandably so. Justifies what was being done & takes as input real observables.

Quoting prices in terms of Black-volatility does not actually mean that you believe in the lognormal model. Cap prices are quoted as “flat volatility”, i.e. the same constant $\gamma$ that when plugged into caplets & summed gives the price.
Another option-type contract is the swaption

\[
\begin{array}{c|c|c|c|c}
\text{today} & \text{swaption expiry} & \text{swap starts} & \text{1st swap cashflow} & \text{swap ends, i.e. last cashflow date} \\
\hline
\begin{array}{c}
t \\
T_i \\
T_n T_{n+1} = T_m + \delta \\
T_n
\end{array}
\end{array}
\]

The time-$T_i$ value of the swaption (i.e. the swaption price at its expiry date) is

\[
\pi^{\text{swopt}}(T_l; T_i, T_m, T_n, \delta, \omega) = \delta (\kappa(T_l; T_m, T_n, \delta) - \omega)^+ \sum_{j=m+1}^{n} P(T_l; T_j).
\]

- Hard to price anything that is not a cap.

- Requires considerable concentration to keep track of all necessary time-indices & integrations.

There is an extensive literature on market models. Nice recent articles by Pelsser, Driessen, de Jong.

Put $X(t) = \delta \sum_{j=m+1}^{n} P(t; T_j)$. This a perfectly legitimate choice of numeraire, so it induces an equivalent martingale measure $Q^X$. Then

\[
\pi^{\text{swopt}}(t; T_l, T_m, T_n, \delta, \omega) = X(t)E^Q_t \left( (\kappa(T_l; T_m, T_n, \delta) - \omega)^+ \right).
\]

and the process $\{\kappa(t; T_m, T_n, \delta)\}_{t}$ is a $Q^X$-martingale.

This, known as the swap-measure approach, can lead to Black-type formulas for swaptions.

A few calculations show that lognormal volatility of swap rates is not consistent with lognormal LIBOR volatility.

So for $t < T_l$, the swaption price can be written as

\[
\pi^{\text{swopt}}(t; \ldots) = P(t; T_l)E^Q_{T_l} \left( \delta (\kappa(T_l; T_m, T_n, \delta) - \omega)^+ \sum_{j=m+1}^{n} P(T_l; T_j) \right).
\]

If $T_l = T_m$ then that we can rewrite the swaption pay-off as

\[
\left( 1 - \sum_{j=m+1}^{n} \alpha_j P(T_m, T_j) \right)^+,
\]

with $\alpha_j = \delta \omega$ for $j \leq n-1$ & $\alpha_n = 1 + \delta \omega$. So the swaption is really a put option on a coupon-bearing bond. The ideas from earlier in the day was used by BGM to derive an approximate swaption-price formula in a lognormal LIBOR market model.
Talk & Chalk

A fun example of caplets(!)

Closing remarks; what to do next?