**RP-exercise, I**

Björk’s exercise 17.7. (OK, I know, that’s lazy on my part, but it is a nice exercise.)

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**RP-exercise, II**

This exercise deals with swap-contracts and the pricing of option contracts with swaps as underlying assets (so-called swaptions). The aim of the exercise is not to derive/state/prove specific formulas, but rather to outline how to build models such that this can be done. “Seeing the structure”, so to say.

Consider an arbitrage-free economy where as usual zero-coupon bond prices are denoted by $P(t; T)$. The simple forward rate, or the $\delta$-forward LIBOR rate (for date $T$ agreed upon at time $t$) is given by

$$L_\delta(t; T) = \frac{P(t; T + \delta) - P(t; T)}{\delta P(t; T + \delta)}.$$  

(If necessary, you should convince yourself that this is a very sensible definition.)

Consider a set of dates (called a tenor-structure) $T_0, T_1, \ldots, T_n$ that are $\delta$-equidistant (i.e. $T_i = T_{i-1} + \delta$), and let $t$ denote the current date.

A swap (or more precisely: a forward payer swap for the specific tenor-structure with fixed rate $\kappa$; evidently that’s unbearable to say more than once) is a contract that pays

$$\delta(L_\delta(T_{i-1}; T_{i-1}) - \kappa)$$

at date $T_i$ for $i = 1, \ldots, n$.

Show that the time-$t$ value of the cash-flows from the swap is

$$\pi^{sw}(t; T_0, T_n, \delta, \kappa) = P(t; T_0) - P(t; T_n) - \delta \kappa \sum_{i=1}^{n} P(t; T_i),$$

thus showing that the value of the swap is independent of volatility despite its future cash-flows being stochastic.

The swap-rate (again for this particular swap) $\omega(t; T_0, T_n, \delta)$ is value of $\kappa$ that solves $\pi^{sw}(t; T_0, T_n, \delta, \kappa) = 0$, i.e.

$$\omega(t; T_0, T_n, \delta) = \frac{P(t; T_0) - P(t; T_n)}{\delta \sum_{i=1}^{n} P(t; T_i)}.$$

Now we consider a swaption. This is the right but not the obligation to enter into swap contact with a specific fixed rate, say $\kappa$, at a (specific) future date. On a time line, things look like this:
Show that the time-\( T_i \) value of the swaption (i.e. the swaption price at its expiry date):

\[
\pi^{\text{swopt}}(T_i; T_i, T_m, T_n, \delta, \kappa) = \delta(\omega(T_i; T_m, T_n, \delta) - \kappa) + \sum_{j=m+1}^{n} P(T_i; T_j) .
\]

Show that for \( t < T_i \), the swaption price is given by

\[
\pi^{\text{swopt}}(t; T_i, T_m, T_n, \delta, \kappa) = P(t; T_i) \mathbb{E}_t^{Q_{T_i}} \left( \delta(\omega(T_i; T_m, T_n, \delta) - \kappa) + \sum_{j=m+1}^{n} P(T_i; T_j) \right) .
\]

where \( \mathbb{E}_t^{Q_{T_i}} \) denotes conditional expectation under the the \( T_i \)-forward measure.

The swaption as an option on a coupon-bearing bond

Assume that \( T_i = T_m \). Show that we can rewrite the swaption pay-off as

\[
\left( 1 - \sum_{j=m+1}^{n} \alpha_j P(T_i; T_j) \right)^+ ,
\]

with \( \alpha_j = \delta \kappa \) for \( j \leq n - 1 \) and \( \alpha_n = 1 + \delta \kappa \), and argue that a swaption is really just a put-option on a coupon-bearing bond. (So last years exam-question gives you an idea how to price it, even though that approach is only directly applicable in a one-factor model.)

Swap-measure approach

Put \( X(t) = \delta \sum_{j=m+1}^{n} P(t; T_j) \). This a perfectly legitimate choice of numeraire, so it induces an equivalent martingale measure \( Q^X \), i.e. a measure such that prices of traded assets are martingales when discounted by \( X \).

Show that

\[
\pi^{\text{swopt}}(t; T_i, T_m, T_n, \delta, \kappa) = X(t) \mathbb{E}_t^{Q^X} \left( (\omega(T_i; T_m, T_n, \delta) - \kappa)^+ \right) .
\]
Show that the process \( \{ \omega(t; T_m, T_n, \delta) \} \) is a \( Q^X \)-martingale. Use this to argue that an arbitrage-free model can be specified as

\[
d\omega(t; T_m, T_n, \delta) = \sigma_{sw}(t; T_m, T_n, \delta) dW^X(t),
\]

where \( \sigma_{sw} \) is a constant (of appropriate dimension). Further, argue that the considered swaption can be priced in closed-form by a Black-Scholes-like formula. (This seems very nice, but beware of the phrase “the considered swaption”. If you take a different swap-rate and specify that as a lognormal martingale under its relevant measure, then that is (very likely to be) in conflict with the first one being lognormal. And LIBOR-rates aren’t lognormal either. However, numerical investigations indicate that discrepancies are small.)