RP-exercise

This exercise deals with pricing of call-options on coupon-bearing bonds (which is what bonds in the real world look like; they’re not zero coupon bonds). Historically, this was where the forward-measure technique first proved its worth. The exercise does contain a lot of text and quite a number of questions, but just take it “one step at the time”.

Suppose that $Y$ solves the stochastic differential equation

$$dY(t) = g(t)Y(t)dt + h(t)Y(t)dW(t),$$

where $W$ is a 1-dimensional Brownian motion, and $g$ and $h$ are deterministic functions. **Show that** for $s \leq t$ we have

$$Y(t) = Y(s) \exp \left( \int_s^t (g(u) - \frac{1}{2} h^2(u)) du + \int_s^t h(u) dW(u) \right),$$

and **conclude that** $Y$ is a lognormal process.

We now consider an arbitrage-free economy where zero-coupon bond prices are driven 1-dimensional Brownian noise, i.e. we can write

$$dP(t, T) = r(t) P(t, T) dt + \sigma_p(t, T) P(t, T) dW^Q(t).$$

In this economy we consider a *coupon bond* that has deterministic payments $\alpha_1, \ldots, \alpha_N$ occurring at dates $T_1, \ldots, T_N$. Clearly the price of this coupon bond is

$$\pi^B(t) = \sum_{i: T_i > t} \alpha_i P(t, T_i).$$

(It is strict inequality, ">", to keep in line with prices being ex-dividend, see Björk chapter 11.) The last ingredient we need is a strike-$K$, expiry-$T$ European call-option on the coupon bond. **Show that** the arbitrage-free price of the call-option, $\pi^C(t)$, can be expressed as

$$\pi^C(t) = \sum_{i: T_i > T} \alpha_i P(t, T_i) Q_i^T (\pi^B(T) > K) - K P(t, T) Q_i^T (\pi^B(T) > K), \quad (1)$$

where $Q_i^T$ denotes $\mathcal{F}_t$-conditional probabilities under the $T$-forward measure. Even with deterministic ZCB-price volatility, $\pi^B$ isn’t lognormal (why?), so it appears we are stuck with (1), at least it seems impossible to express the price as a Black-Scholes type-expression (i.e. where the $Q$’s have become the normal-distribution function $\Phi$ at a suitable point.) The rest of the exercise will help you show that “it ain’t necessarily so.”
We start by putting more structure on the ZCB-volatility by assuming that it is deterministic and of a special functional form. More specifically suppose that there exists "nice" deterministic functions $g$ and $h$ such that we can write

$$\sigma_F(t, T) = (g(T) - g(t))h(t).$$

This class includes the Vasicek model and corresponds (by which result in Björk?) to a forward rate volatility that is multiplicatively separable in $t$ and $T$, specifically $\sigma_F(t, T) = -g'(T)h(t)$. We assume that $g$ is monotone. (Otherwise we get well-definedness problems.)

Let us now focus on one of the terms in (1), $\mathbb{Q}_t^T(\pi^B(T) > K)$ for instance. (The others can be treated in the exact same way as we are about to.) For each $T_i$ define the process $Z(\cdot, T, T_i)$ by $Z(t, T, T_i) = P(t, T_i)/P(t, T)$. Show that

$$dZ(t, T, T_i) = (g(T_i) - g(T))h(t)W^T(t),$$

where $W^T$ is a Brownian motion under the $T$-forward measure.

**Conclude that** we may write

$$P(T, T_i) = \frac{P(t, T_i)}{P(t, T)} \exp \left( -\frac{1}{2}(g(T_i) - g(T))^2 \int_t^T h^2(u)\,du + (g(T_i) - g(T)) \int_t^T h(u)W^T(u) \right).$$

The first term inside “exp” is deterministic. The important observation is that the stochastic part of the second term is the same for all $T_i$. Use these observations to **conclude that** if we put $H(t, T) = \int_t^T h^2(u)\,du$ we may write

$$\pi^B(T) = \pi^B(T; x)_{x=n(0,1)} = \sum_{i:T_i>T} \text{"some function"}(t, T, T_i) \times \exp \left( (g(T_i) - g(T))\sqrt{H(t, T)} \times x \right),$$

where $n(0,1)$ means a variable that is $N(0,1)$-distributed under $\mathbb{Q}^T$ and is independent of $\mathcal{F}_t$.

**Conclude that** $x \mapsto \pi^B(T; x)$ is monotone and that there exists a unique $d(t, T, T_i)$ such that $\pi^B(T; -d(t, T, T_i)) = K$ and that we may write

$$\mathbb{Q}_t^T(\pi^B(T) > K) = \Phi(d(t, T, T_i)).$$

**Convince yourself (and me) that** this machinery works for all the terms in (1) and that you get

$$\pi^C(t) = \sum_{i:T_i>T} \alpha_i P(t, T_i) \Phi(d_{T_i}) - KP(t, T)\Phi(d_T),$$

where some of the notational dependencies in the $d$’s have been suppressed, but they can all be found by solving equations that are easily dealt with numerically.