Multi-dimensional Term Structure Models

We will focus on the affine class. But first some motivation/guide to the literature.

A generic one-dimensional model for zero-coupon yields, \( y(t; \tau) \), looks like this

\[
dy(t; \tau) = \ldots dt + \sigma_y(t, \tau)dW,
\]

where \( W \) is a one-dimensional Brownian motion (under some measure).

This means that

\[
\text{cov}_t(\Delta y(t, \tau_i), \Delta y(t, \tau_j)) = \sigma_y(t, \tau_i)\sigma_y(t, \tau_j)\Delta t + O(\Delta t^2),
\]

where \( \Delta y(t, \tau_i) = y(t + \Delta t, \tau_i) - y(t, \tau_i) \).
So

\[
\frac{\text{cov}_t(\Delta y(t, \tau_i) \Delta y(t, \tau_j))}{\sqrt{\text{var}_t(\Delta y(t, \tau_i))} \sqrt{\text{var}_t(\Delta y(t, \tau_j))}} \rightarrow 1 \quad \text{for } \Delta t \rightarrow 0,
\]

or in words (instantaneous) yield(-change)s are (conditionally) perfectly correlated.
How does this look for US data?

<table>
<thead>
<tr>
<th>$(\tau_i, \tau_j)$</th>
<th>0.25</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.0175</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.9519</td>
<td>0.0174</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.8628</td>
<td>0.9538</td>
<td>0.0168</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.7716</td>
<td>0.8773</td>
<td>0.9557</td>
<td>0.0153</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.6231</td>
<td>0.7399</td>
<td>0.8433</td>
<td>0.9306</td>
<td>0.0124</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.5160</td>
<td>0.6285</td>
<td>0.7205</td>
<td>0.8205</td>
<td>0.9335</td>
<td>0.0102</td>
</tr>
</tbody>
</table>

(Monthly data ($\Delta = 1/12$) from 1952 to 2005. Diagonal elements are standard deviations, off-diagonals are correlations.)

Off-diagonal elements do not look too much like 1’s. (Not a formal test, and you could ask is $\Delta = 1/12$ is small, but still . . .)
Factor Analysis

A classical statistical discipline. Not a formal model, but healthy for data analysis.

Ask: What is the effective rank of the covariance matrix? A sensible measure of this is “how many eigenvalues are (close to) 0?”

And you can answer this question in terms explained variance (cumulative sum divided by total sum of eigenvalues).

Here:

<table>
<thead>
<tr>
<th>up to eigenvector #</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>fraction of variance explained</td>
<td>0.869</td>
<td>0.969</td>
<td>0.988</td>
<td>0.995</td>
<td>0.997</td>
<td>1.000</td>
</tr>
</tbody>
</table>

and we’d say something like “2-3 factors”, probably.
Further, there is a good chance you’ll hear the words *level, slope and curvature* and see graphs like this one
Affine Factor Models

An $n$-factor model is one where

$$ r(t) = R(X(t)), $$
$$ dX(t) = \mu(X(t))dt + \Sigma(X(t))dW^Q(t), $$

where $X$ is a stochastic process whose coordinates are referred to as factors (abstract so far), and $R : \mathbb{R}^n \mapsto \mathbb{R}$, $\mu : \mathbb{R}^n \mapsto \mathbb{R}^n$, and $\Sigma : \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$ are functions.

We say that the model is affine if

$$ R, \text{ each coordinate in } \mu, \text{ and each coordinate in } \Sigma \Sigma^\top $$

are affine functions (of some $x \in \mathbb{R}^n$). (Note that in general $[\Sigma \Sigma^\top]_{i,j} \neq [\Sigma_{i,j}]^2$.)
Or with symbols

\[
R(x) = \delta_0 + \delta^\top x
\]

\[
\mu(x) = K(\theta - x)
\]

\[
\Sigma(x) = \sum_{n \times n} \sqrt{S(x)},
\]

where everything that looks constant is, and \(S(x)\) denotes a diagonal matrix whose \(i\)th diagonal element is

\[
[S(x)]_{i,i} = \alpha_i + \beta_i^\top x,
\]

and the meaning of matrix-\(\sqrt{\cdot}\) is then obvious. (As is some positivity restriction.)
This is the most common parametrization. “Too easy” if it were the only one!

Clear that $R$ and $\mu$ are affine.
Not hard to convince yourself that $(\Sigma S(x)\Sigma^\top)_{i,j}$ is affine in $x$.
Less clear that the form of the volatility isn’t a restriction. Duffie & Kan show that it isn’t really.

And now for the zero coupon bond pricing.

We (the first equality: always, the second: here) have

$$P(t, T) = \mathbb{E}_t^Q \left( \exp - \int_t^T r(u) du \right) = \mathbb{E}_t^Q \left( \exp - \int_t^T (\delta_0 + \sum_{i=1}^n \delta_i X_i(u)) du \right),$$
Put differently $P(t, T) \exp(-\int_0^t r(u)du)$ is a $Q$-martingale.

And that can be only if the $Q$-drift (rate) of the ZCB price is $r(t) = R(X(t))$.

Because $r(t) = R(X(t))$ and $X$ is a time-homogeneous Markov process, $P(t, T)$ is of the form

$$P(t, T) = f(X(t), T-t),$$

for some function $f$.

This is heading towards a restriction on the drift.
Affine ZCB Price Theorem (Duffie & Kan)

In an affine model, ZCB prices are of the exponentially affine form

\[ P(t, T) = \exp(A(T - t) - B^\top (T - t) X(t)) \]

where the function \( B : \mathbb{R} \mapsto \mathbb{R}^n \) solves the system of ODEs

\[ \frac{dB}{d\tau} = \delta - K^\top B(\tau) - \frac{1}{2} \sum_{i=1}^n (\Sigma^\top B(\tau) i^2 \beta_i \]

and the function \( A : \mathbb{R} \mapsto \mathbb{R} \) solves the ODE

\[ \frac{dA}{d\tau} = -\delta_0 - \theta^\top K^\top B(\tau) + \frac{1}{2} \sum_{i=1}^n (\Sigma^\top B(\tau) i^2 \alpha_i. \]
**Proof:** Use

- “Martingality”.
- Multi-dimensional Ito; careful w/ vectors and matrices.
- The matching principle: If $a + b^T x = 0$ for all $x$ (in some open set), then $a = 0$ and $b = 0$.

**BLACKBOARD**
Remarks

- For $B$ we have a coupled system of $n$ ODEs. Much better than PDEs w/ multidimensional state variables.

- Sometimes we can solve in closed form, sometimes we can’t. Not hard numerically (“pedestrian” Euler, or Runge/Kutta).

- Two theoretical issues (Duffie & Kan)
  - *The converse.* (Proposition p. 386.) Note “a non-degeneracy condition”.
  - Well-definedness (admissibility) when some $\beta$’s are non-0. (Theorem p. 388.)

- Two examples: Gaussian models and “sum of independent CIRs”. Answer some questions, raise others $\rightarrow$ Dai & Singleton.