Empirically: Variations in “the” short rate “explains” “a large percentage” of the “variation” of the whole term structure. (“The tail wagging the dog”)

An model where only real-world short rate dynamics are specified is not complete. Just think in terms of # traded assets and # sources of risk.

We get consistency relation between ZCBs of different maturities. (So nice we may forget we have a “problem” at all.)

We extend the usual PDE derivation to stochastic interest rates. A line of reasoning first done in Vasicek(1977).
Look (first) at the case where

\[ dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW^\mathbb{P}(t) \]

where \( \mu \) and \( \sigma \) are functions and \( W^\mathbb{P} \) is a 1-dimensional Brownian motion under the real-world probability measure \( \mathbb{P} \). So \( r \) is Markov wrt. its own filtration.

Suppose all kinds of ZCB exist. The “formal” equation

\[ P(t; T) = \mathbb{E}_t^\mathbb{Q} \left( \exp \left( - \int_t^T r(u)du \right) \right) \]

makes us conjecture that

\[ P(t; T) = F(t, r(t); T) \]
for some smooth function $F$ (of 3 variables.)

Hide $T$-dependence in a superscript and use Ito to get

$$
\frac{dF_T}{F_T} = \left( \frac{F_T}{F_T} \right) F_T \frac{t}{F_T} + \mu F_T \frac{T}{F_T} + \frac{1}{2} \sigma^2 F_T \frac{T}{F_T} dt + \frac{\sigma F_T}{F_T} dW_P(t)
$$

Now make a self-financing portfolio with a $T$-ZCB and an $S$-ZCB. $V$ is the value process and $(u_T, u_S)$ relative portfolio weights. From Chapter 6 we have

$$
\frac{dV}{V} = u_T \frac{dF_T}{F_T} + u_S \frac{dF_S}{F_S}
$$

$$
= (u_T \alpha_T + u_S \alpha_S) dt + (u_T \sigma_T + u_S \sigma_S) dW_P(t)
$$
By construction we must have $u_T + u_S = 1$, but still 1 degree of freedom. A clever choice is $u_T \sigma^T + u_S \sigma^S = 0$.

The $dW^P$-term vanishes, we get

$$u_T = \frac{-\sigma^S}{\sigma^T - \sigma^S}$$

(symmetric in $S$), and

$$\frac{dV}{V} = \frac{\alpha^S \sigma^T - \alpha^T \sigma^S}{\sigma^T - \sigma^S} \ dt.$$

must $= r(t)$ otherwise arbitrage
Rewriting

\[
\frac{\alpha^S - r(t)}{\sigma^S} = \frac{\alpha^T - r(t)}{\sigma^T}
\]

LHS doesn’t depend on \( T \), RHS doesn’t depend on \( S \) \( \Rightarrow \) the ratio is independent of maturity:

\[
\frac{\alpha^S - r(t)}{\sigma^S} := \lambda(r(t); t).
\]

\( \lambda \): “market price of risk”; interpretation as excess expected return relative to volatility. Has to be exogenously specified.

Usually: Postulate form that gives same structure under \( \mathbb{P} \) and \( \mathbb{Q} \) — a subtlety that people may be obtuse about.
Substitute back and get the term structure PDE:

\[ F_t^T + (\mu - \lambda \sigma) F_r^T + \frac{1}{2} \sigma^2 F_{rr}^T = r F^T \quad \text{and} \quad F^T(T; r) = 1 \]

This may be Feynman-Kac represented and we may change measure:

\[ F(t, r(t); T) = \mathbb{E}^Q \left( \exp \left( - \int_t^T r(s) ds \right) \right) \]

where

\[ dr(s) = (\mu - \lambda \sigma) ds + \sigma dW_Q^Q(s) \]

Note: Clearly \( P(t; T)/\beta(t) \) is a \( Q \)-martingale.
Writing

\[
\frac{dP(t; T)}{P(t; T)} = \alpha^T(t; T)dt + \sigma^T(t, T)dW^P = r(t)dt + \sigma^T(t, T) \left( dW^P + \frac{\alpha^T - r(t)}{\sigma^T} dt \right)
\]

= \text{d}W^Q, \text{ by Girsanov}

shows that pieces fit.

We’re still not very concrete.